TIME SERIES: FINAL EXAM

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ABSTRACT. This document contains problems and solutions for the final exam of the Time Series course at Universidad Torcuato Di Tella, first trimester 2025. Datasets were provided as EViews workfiles (`.wf1`) and converted to csv format for use with alternative software. The code used to solve each problem is attached, as a separate script for each problem, and can also be found at `https://github.com/felipetappata/series-final`. References, a list of figures, and a list of tables are included at the end of the document.

1. Stock Prices and Dividends

Problem 1. Use the EViews data file `Shiller2018a.wf1`. Consider the variables `dlogrdiv` (real dividends growth) and `dlogrsp` (real stock prices growth).

(a) Find each variable's ARMA(p, q) representation. Test whether the series has ARCH effects.
(b) Whenever you find GARCH effects, estimate a model that accounts for those effects.
(c) Explain how you have chosen your preferred model.
(d) Repeat (a), (b) and (c) for the sub-sample 1900-1950 and 1951-2018. Have your results changed? Using the information of (d), reinterpret the results of (a), (b) and (c). Comment on the results.

Solution 1 (a). The so-called "Box-Jenkins modeling philosophy" for forecasting is described in Hamilton (1994, p. 110) as follows:

(1) Transform the data, if necessary, so that the assumption of covariance-stationarity is a reasonable one.
(2) Make an initial guess of small values for p and q for an ARMA(p, q) model that might describe the transformed series.
(3) Estimate the parameters in $\phi(L)$ and $\theta(L)$.
(4) Perform diagnostic analysis to confirm that the model is indeed consistent with the observed features of the data.

Given that both of the variables with which we are concerned represent growth rates, the assumption of covariance-stationarity does not seem unreasonable. In order to make guesses for p and q, we will examine the sample autocorrelation function (ACF) and the sample partial autocorrelation function (PACF) of each variable. Figure 1 displays these functions for both `dlogrdiv` (real dividends growth) and `dlogrsp` (real stock prices growth).

To perform diagnostic analysis for the estimated model amounts in our case to examining the sample ACF of the residuals. This can be done by starting with a simple model and increasing the number of parameters until the sample ACF of the residuals is consistent with white noise. We estimate an array of ARMA(p, q) models for each variable, ranging from

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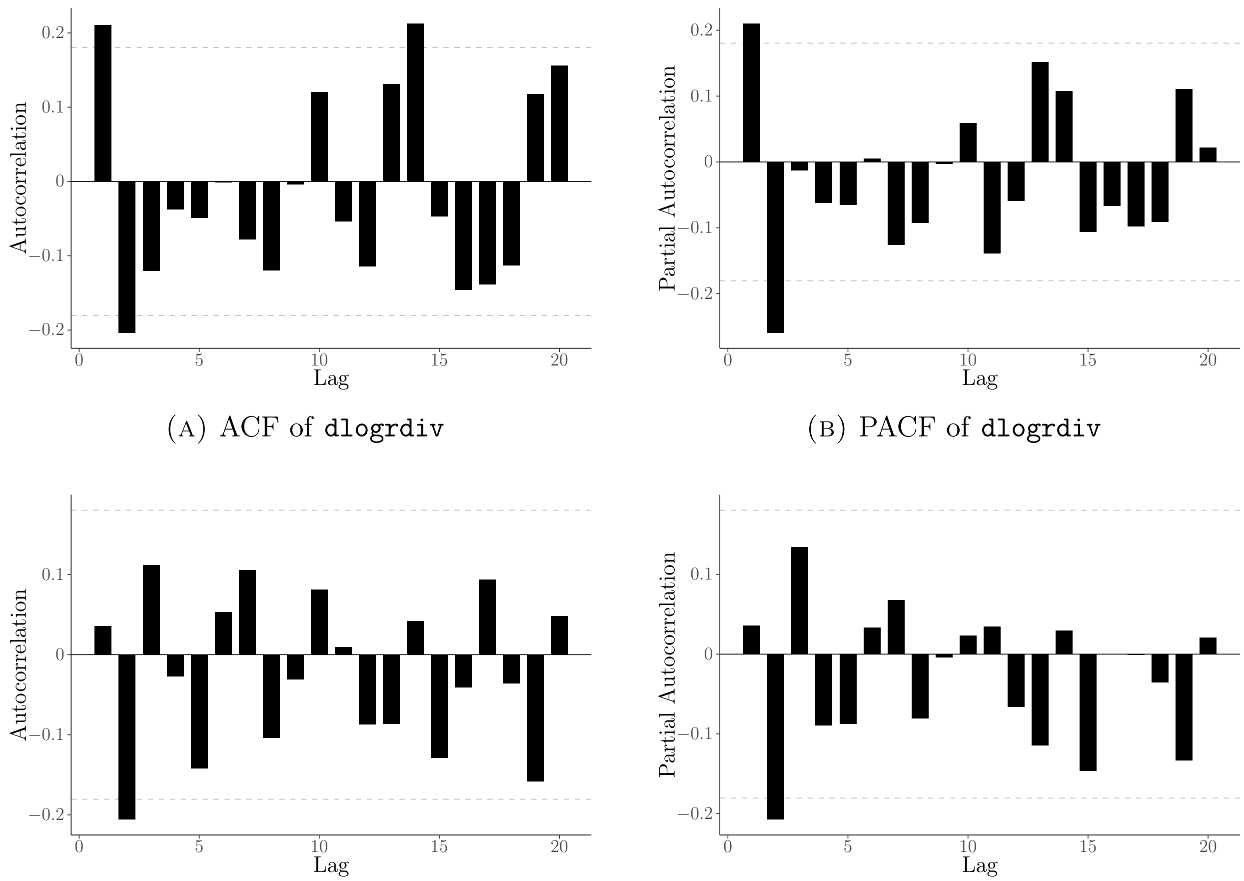


(A) ACF of `dlogrdiv` (B) PACF of `dlogrdiv`

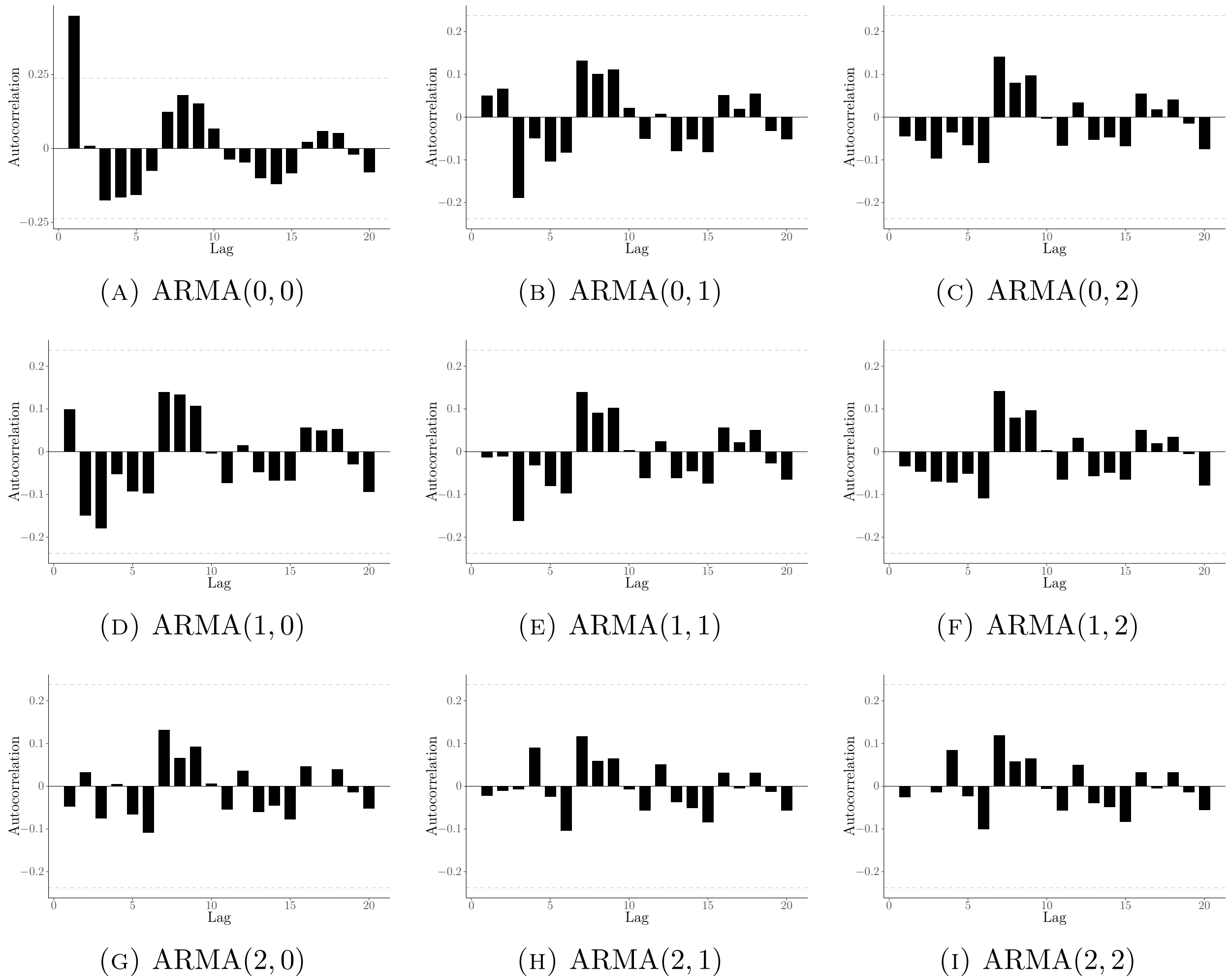
(C) ACF of `dlogrsp` (D) PACF of `dlogrsp`

FIGURE 1. Sample autocorrelation and partial autocorrelation functions for real dividends growth and real stock prices growth

$p = 0$ to $p = 2$ and $q = 0$ to $q = 2$. Of course, the model ARMA$(0, 0)$ is simply the sample mean of the series for which it is estimated—a somewhat trivial model, but we include it because it is simpler to program the iterative procedures without removing it. The sample ACF of the residuals of each model for `dlogrdiv` can be seen in Figure 2. The sample ACF of the residuals of each model for `dlogrsp` can be seen in Figure 3.

In order to select the best model, we calculate the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC, also known as Scharz Criterion) for each model. Tables 1 and 2 show the AIC values for the ARMA(p, q) models estimated for `dlogrdiv` and `dlogrsp`, respectively. Tables 3 and 4 show the BIC values for the same models.

Upon inspecting the residuals in Figure 2, we may restrict our attention to models ARMA$(2, 0)$, ARMA$(2, 1)$, ARMA$(2, 2)$, ARMA$(0, 2)$, and ARMA$(1, 2)$ for `dlogrdiv` and to models ARMA$(2, 0)$, ARMA$(2, 1)$, ARMA$(2, 2)$, ARMA$(0, 2)$, and ARMA$(1, 2)$ for `dlogrdiv` because they are the only ones which produce a more or less "clean" sample ACF of the residuals. Inspecting the AIC and BIC values in Tables 1 and 2, we see that the ARMA$(2, 0)$ model exhibits some of the lowest AIC and BIC values among the models we are considering. The model chosen for `dlogrdiv` is ARMA$(2, 0)$, although ARMA$(1, 2)$ would not be unreasonable.



(A) ARMA$(0,0)$ (B) ARMA$(0,1)$ (C) ARMA$(0,2)$

(D) ARMA$(1,0)$ (E) ARMA$(1,1)$ (F) ARMA$(1,2)$

(G) ARMA$(2,0)$ (H) ARMA$(2,1)$ (I) ARMA$(2,2)$

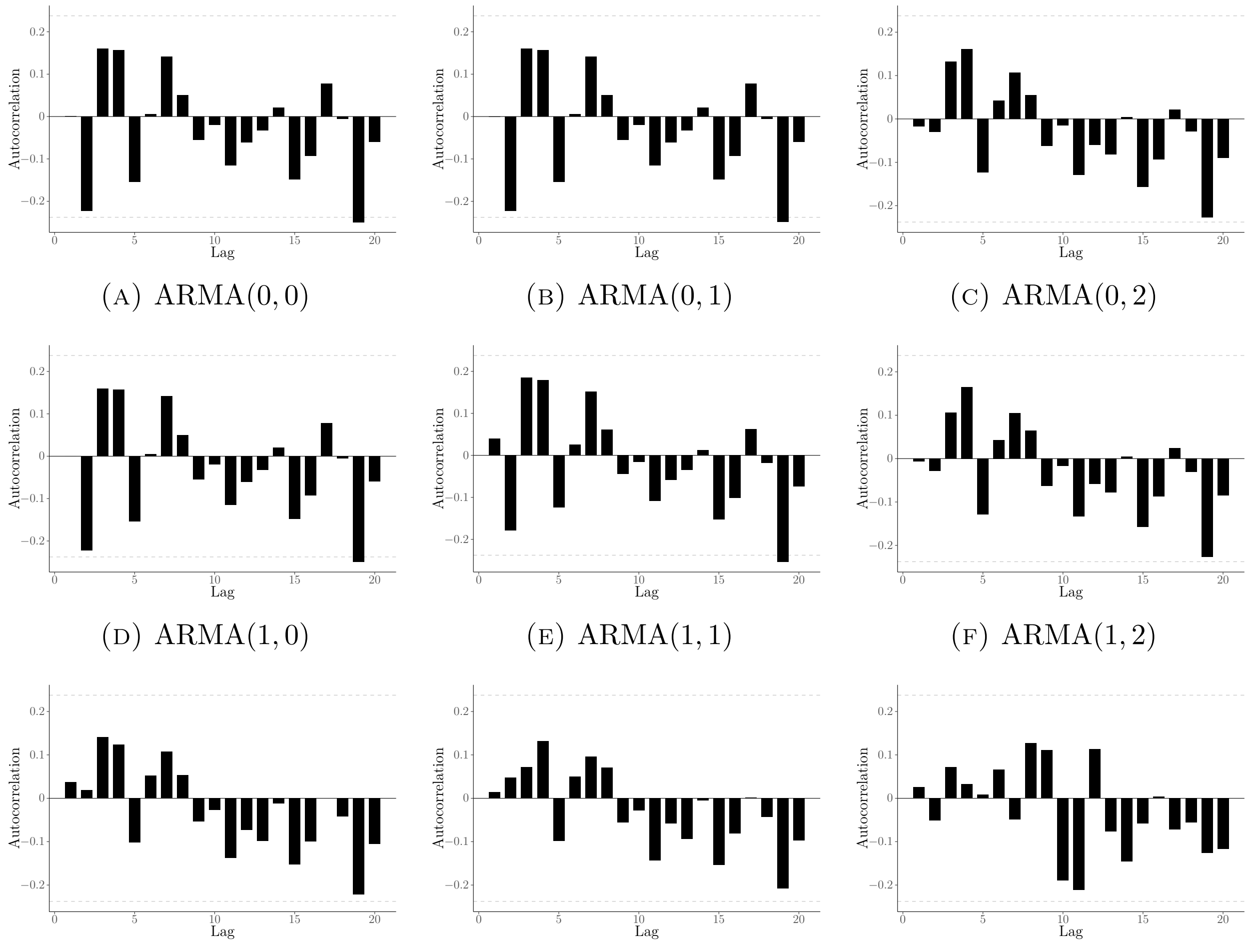
FIGURE 2. Sample autocorrelation functions of residuals for ARMA(p,q) models estimated for `dlogrdiv` (real dividends growth)

TABLE 1. AIC values for ARMA(p,q) models estimated for `dlogrdiv` (real dividend growth)

	$q=0$	$q=1$	$q=2$
$p=0$	-180.41	-187.43	-188.19
$p=1$	-183.71	-186.75	-190.94
$p=2$	-189.83	-187.86	-189.07

Inspection of the residuals in Figure 3 shows that we may eliminate from consideration the models ARMA$(0,0)$, ARMA$(0,1)$, and ARMA$(1,0)$, as they produce large autocorrelation in residuals. Among the remaining models, ARMA$(0,2)$ is the unanimous choice according to the AIC and BIC values in Tables 1 and 2. The estimated ARMA$(2,0)$ model for `dlogrdiv` is shown in Table 5.

The estimated ARMA$(0,2)$ model for `dlogrsp` is shown in Table 6.



(A) ARMA$(0,0)$ (B) ARMA$(0,1)$ (C) ARMA$(0,2)$

(D) ARMA$(1,0)$ (E) ARMA$(1,1)$ (F) ARMA$(1,2)$

(G) ARMA$(2,0)$ (H) ARMA$(2,1)$ (I) ARMA$(2,2)$

FIGURE 3. Sample autocorrelation functions of residuals for ARMA(p,q) models estimated for `dlogrsp` (real stock prices growth)

TABLE 2. AIC values for ARMA(p,q) models estimated for `dlogrsp` (real stock prices growth)

	$q=0$	$q=1$	$q=2$
$p=0$	-52.64	-50.89	-55.19
$p=1$	-50.78	-52.83	-53.65
$p=2$	-53.93	-54.13	-53.06

In symbols, we have assumed that the processes followed by the variables are given by

$$\texttt{dlogrdiv}_t = c + \phi_1 \texttt{dlogrdiv}_{t-1} + \phi_2 \texttt{dlogrdiv}_{t-2} + u_t, \tag{1}$$
$$\texttt{dlogrsp}_t = \mu + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \varepsilon_t, \tag{2}$$

TABLE 3. BIC values for ARMA(p, q) models estimated for `dlogrdiv` (real dividend growth)

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-174.87	-179.11	-177.11
$p = 1$	-175.40	-175.67	-177.09
$p = 2$	-178.74	-174.01	-172.45

TABLE 4. BIC values for ARMA(p, q) models estimated for `dlogrsp` (real stock prices growth)

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-47.09	-42.58	-44.11
$p = 1$	-42.47	-41.75	-39.80
$p = 2$	-42.85	-40.27	-36.44

TABLE 5. ARMA$(2, 0)$ Model for Real Dividend Growth (`dlogrdiv`)

	Dependent variable:
	`dlogrdiv`
AR(1)	0.2631^{***}
	(0.0887)
AR(2)	-0.2562^{***}
	(0.0883)
Intercept	0.0144
	(0.0097)
Observations	118
σ^2	0.0109
Akaike Inf. Crit.	-189.8258
Note:	*p<0.1; **p<0.05; ***p<0.01

and estimated the parameters to arrive at

$$\texttt{dlogrdiv}_t = 0.01 + 0.26\texttt{dlogrdiv}_{t-1} - 0.26\texttt{dlogrdiv}_{t-2} + \hat{u}_t, \tag{3}$$

$$\texttt{dlogrsp}_t = 0.02 + 0.09\varepsilon_{t-1} - 0.23\varepsilon_{t-2} + \hat{\varepsilon}_t. \tag{4}$$

We now turn to an examination of the presence of autoregressive conditional heteroskedasticity (ARCH) effects in the residuals of the models we have chosen. In other words, we want to see whether "volatility clustering" is present. The test we will use the Lagrange Multiplier

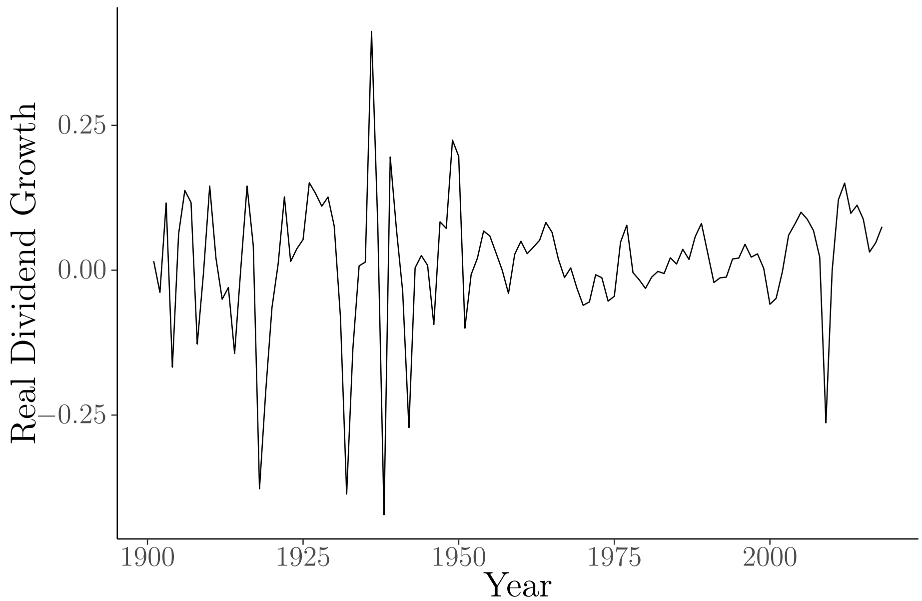
Table 6. ARMA$(0,2)$ Model for Real Stock Price Growth (`dlogrsp`)

	Dependent variable:
	`dlogrsp`
MA(1)	0.0913
	(0.0893)
MA(2)	-0.2325^{***}
	(0.0889)
Intercept	0.0204
	(0.0147)
Observations	118
σ^2	0.0342
Akaike Inf. Crit.	-55.1896
Note:	*p<0.1; **p<0.05; ***p<0.01

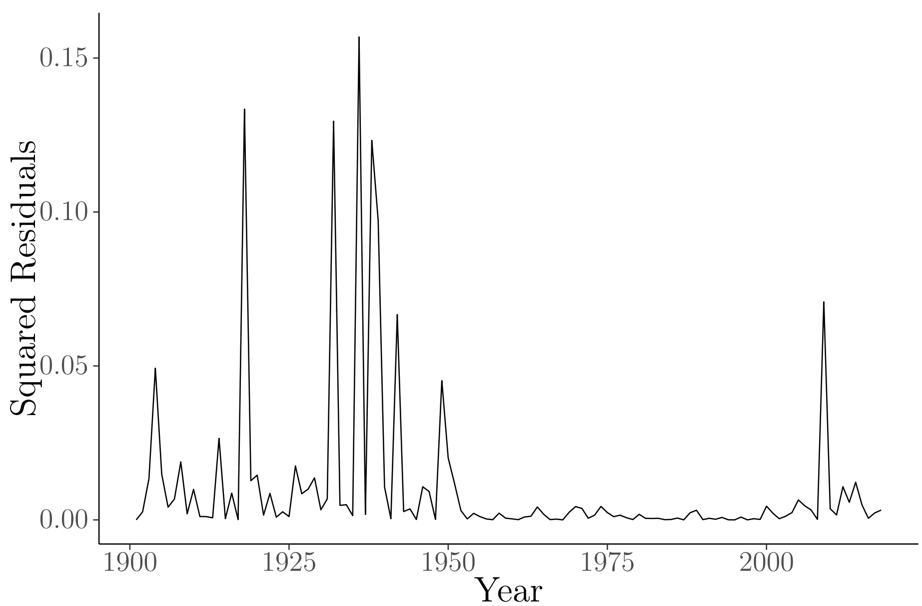
(LM) test of Engle (1982), which tests the null hypothesis that the coefficients of the first q lags of the squared residuals are all zero (in a regression of the squared residuals on the first q lags of the squared residuals). The results of this test for the residuals of the ARMA$(2,0)$ model estimated for `dlogrdiv` are shown in Table 7, for different values of q. As we can see by the low p-values for nearly every specification of the test, there is evidence suggesting the presence of ARCH effects in the model for dividend growth. The presence of "volatility clustering" can be clearly seen in Figure 4 which shows, alongside the series `dlogrdiv` itself, the squared residuals of the ARMA$(2,0)$ model. There is a clear difference in the volatility over the period 1900-1950 and the period 1951-2018, as problem 1 (d) suggests.

The results of the LM test for the residuals of the ARMA$(0,2)$ model estimated for `dlogrsp` are shown in Table 8. For these tests, the p-values are high, implying that the null hypothesis of no ARCH effects cannot be rejected. Comparison of Figure 5, which shows the series `dlogrsp` alongside the squared residuals of the ARMA$(0,2)$ model, with Figure 4 shows the difference between the behavior of volatility in the two series. There do seem to be "spikes" in the squared residuals, but comparison of, for instance, the volatility near 1930 with that near 1975 suggests that one may have difficulty predicting conditional variances using the squared residuals. We may therefore conclude that ARCH effects are present in the residuals of the ARMA$(2,0)$ model for `dlogrdiv` but not in the residuals of the ARMA$(0,2)$ model for `dlogrsp`.

Solution 1 (b). The task at hand is now to find a model that adequately accounts for conditional heteroskedasticity in the residuals of the ARMA$(2,0)$ model for `dlogrdiv`, given the presence of ARCH effects found in solution 1 (a). We will estimate an array of GARCH(p,q) models of the type developed by Bollerslev (1986) and select based on the AIC and BIC values, with a preference for parsimonious models. Table 9 shows the AIC values for the GARCH(p,q) models estimated for `dlogrdiv` (real dividends growth). Table 10 shows the BIC values for the same models. Consideration of AIC and BIC values both suggest that



(A) Real dividends growth (`dlogrdiv`)



(B) Squared residuals of the ARMA$(2, 0)$ model for `dlogrdiv`

FIGURE 4. Evidence of volatility clustering: real dividends growth and squared residuals of the ARMA$(2, 0)$ model for `dlogrdiv`

GARCH$(1, 1)$ is the best model for `dlogrdiv`. We therefore estimate a GARCH$(1, 1)$ model for `dlogrdiv`, with a Student's t-distribution, the results of which are shown in Table 11. The reason for choosing the t-distribution rather than the default Gaussian is that there is evidence of fat tails. This decision is motivated by the results of a likelihood ratio test, as

Table 7. ARCH LM-test results for `dlogrdiv` (real dividends growth) residuals of the ARMA$(2, 0)$ model

Lags	χ^2	p-value
1	0.8185	0.3656
2	4.4095	0.1103
3	6.6683	0.0833
4	15.7059	0.0034
5	17.8160	0.0032
6	19.8053	0.0030
7	19.8050	0.0060
8	20.8246	0.0076
9	20.7986	0.0136
10	21.0609	0.0207
11	21.3944	0.0295
12	21.1296	0.0485
13	20.8339	0.0763
14	24.4753	0.0401
15	24.5065	0.0570
16	25.9707	0.0544
17	27.3502	0.0531
18	35.7803	0.0075
19	35.5456	0.0120
20	35.5243	0.0175

well as comparison of the AIC and BIC values for the two models.[1] It is also supported by the `shape` parameter in Table 11, which suggests that the distribution of the residuals is indeed leptokurtic.

Solution 1 (c). As mentioned briefly in Solution 1 (b), there are several reasons for preferring the GARCH$(1, 1)$ model with Student's t-distribution over others: it is parsimonious and preferred by the information criteria, and it is special among the rest in the sense that it is the simplest GARCH model; that is, the model with fewest parameters that models conditional heteroskedasticity based both on past squared residuals and past conditional variances. It is often used in the literature (for instance, in the context of the CAPM, by Bollerslev, Engle, and Wooldridge, 1988), and has in broad cases been found to outperform more sophisticated models, as in Hansen and Lunde (2005). An argument can also be made that while economic theory might support a relationship between a dividend growth volatility in a given year and that in the previous year, an additional dependence on volatility two years back is more dubious.

Throughout, we have only considered what is often called the standard GARCH (sGARCH) model, which follows the specification of Bollerslev (1986), but we might have had reason to broaden our scope to GARCH-in-mean (GARCH-M) models or Exponential GARCH

[1] These results are not shown but can be easily replicated by estimating the GARCH$(1, 1)$ model with a Gaussian distribution and comparing the results with those of the Student's t-distribution.

Table 8. ARCH LM-test results for `dlogrsp` (real stock prices growth) residuals of the ARMA$(0,2)$ model

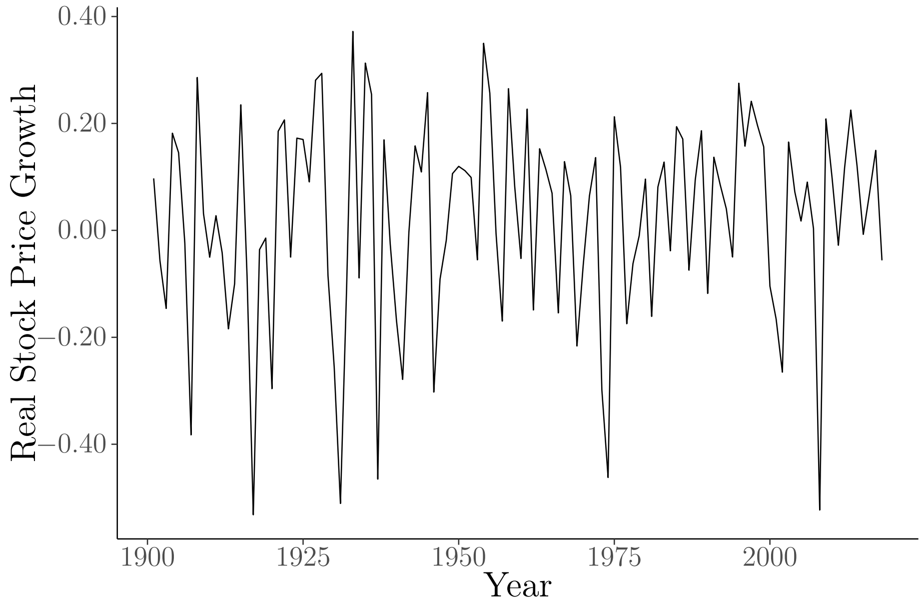
Lags	χ^2	p-value
1	0.0948	0.7582
2	0.8387	0.6575
3	0.9520	0.8129
4	2.4576	0.6522
5	3.5359	0.6180
6	3.9318	0.6859
7	4.0809	0.7704
8	3.9566	0.8610
9	3.8471	0.9212
10	4.2745	0.9341
11	5.9431	0.8771
12	6.5661	0.8849
13	6.8482	0.9098
14	7.1985	0.9268
15	7.7008	0.9352
16	7.6244	0.9593
17	8.8277	0.9455
18	9.8187	0.9377
19	9.8006	0.9577
20	12.6449	0.8921

Table 9. AIC values for GARCH(p,q) models estimated for `dlogrdiv` (real dividends growth)

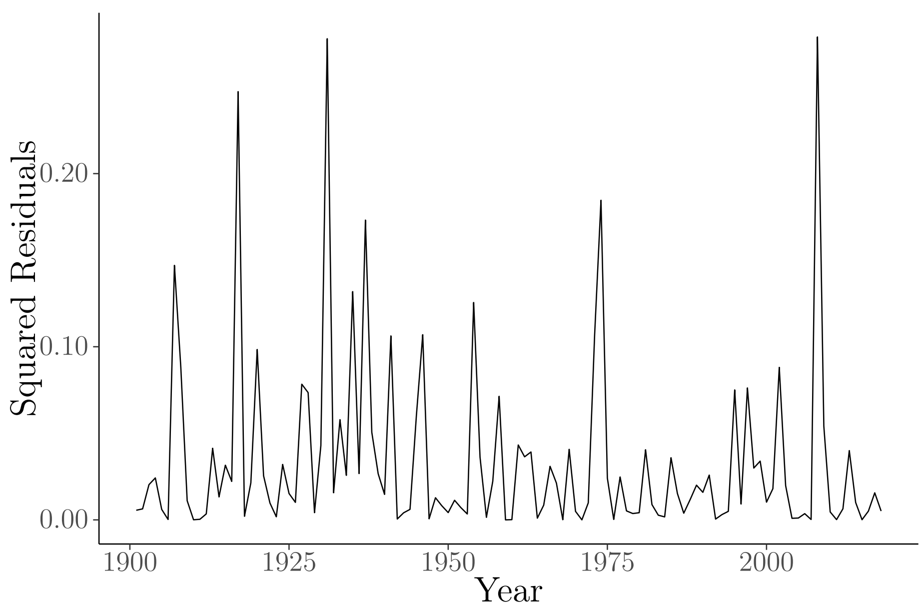
	$q=1$	$q=2$
$p=1$	-2.2290	-2.2127
$p=2$	-2.2120	-2.1958

Table 10. BIC values for GARCH(p,q) models estimated for `dlogrdiv` (real dividends growth)

	$q=1$	$q=2$
$p=1$	-2.0646	-2.0249
$p=2$	-2.0242	-1.9845



(A) Real stock prices growth (`dlogrsp`)



(B) Squared residuals of the ARMA$(0, 2)$ model for `dlogrsp`

FIGURE 5. No strong evidence of volatility clustering: real stock prices growth and squared residuals of the ARMA$(0, 2)$ model for `dlogrsp`

(EGARCH) models, or even Threshold GARCH (or GJR-GARCH) models.[2] The first alternative type mentioned, GARCH-M, allows the conditional mean to depend on the conditional variance, which might be useful in the context of asset pricing, given the theoretical and empirical result referred to as the "risk-return tradeoff." It is not of particular interest

[2]The following comments are based on the introduction in D. B. Nelson (1991). A review of empirical properties and stylized facts of asset returns can be found in Cont (2001).

Table 11. Estimated GARCH$(1,1)$ model for `dlogrdiv` (real dividends growth)

Dependent Variable: `dlogrdiv`			
GARCH Model: sGARCH$(1,1)$			
Mean Model: ARFIMA$(2,0,0)$			
Distribution: Student's t			
Parameter	**Estimate**	**Std. Error**	p**-value**
μ	0.0189$*$	0.0104	0.0686
ϕ_1	0.5712$***$	0.1280	0.0000
ϕ_2	$-0.1964***$	0.0696	0.0048
ω	0.0003	0.0002	0.1887
α_1	0.2837$**$	0.1234	0.0215
β_1	0.7153$***$	0.0918	0.0000
`shape`	3.2799$***$	0.7406	0.0000
Log-likelihood	138.51		
AIC	-2.2290		
BIC	-2.0646		
Observations	118		

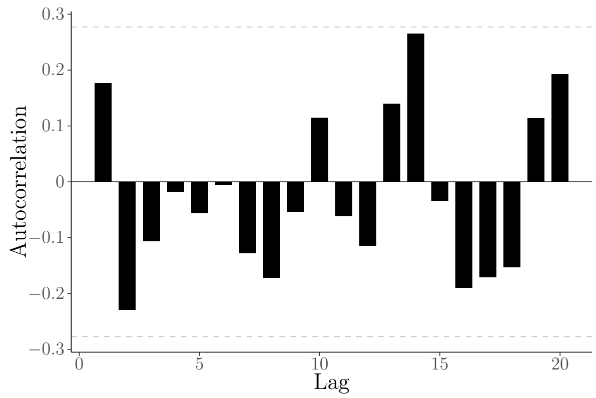
Note: Robust standard errors reported.
*** $p < 0.01$, ** $p < 0.05$, * $p < 0.10$.

in our case to examine this relationship, if it were present, and the fact that we have series for only one asset means that we would be comparing investor attitudes towards risk over time, rather than across assets at given moments, which might be questionable (of course, it might well be justified; the principal point is that we do not wish to model this relationship explicitly). The latter two alternative classes of GARCH model mentioned, EGARCH and GJR-GARCH, are both useful in the context of modeling asymmetric (with respect to the sign of variations) effects of volatility. In the context of models of asset pricing, this is often referred to as the "leverage effect:" the negative association between returns and volatility. As with the GARCH-M model, these alternative classes permit the capturing of dynamics that for the present problem are not of interest. Therefore, as we are studying a relatively simple dataset, without the explicit intention of capturing the risk-return tradeoff or the leverage effect, we will not consider these alternative classes of the GARCH model.

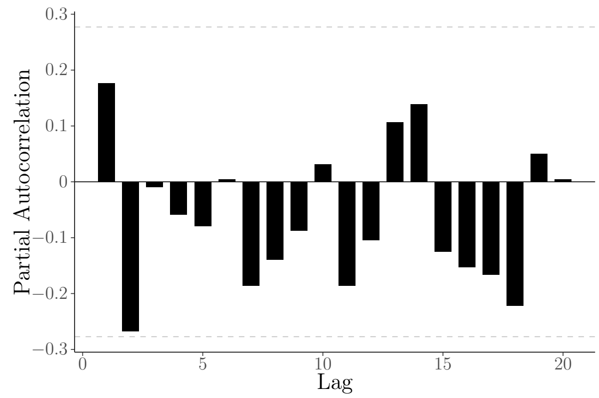
Summing up, the argument that leads one to prefer the GARCH$(1,1)$ model with Student's t-distribution is that it is simple and takes a conservative position in regard to overfitting, is preferred by the AIC and BIC, and is widely used in the literature.

Solution 1 (d). In solution 1 (a), when looking for ARCH effects, we created a plot of the squared residuals of the ARMA$(2,0)$ model for `dlogrdiv` (real dividends growth) alongside the series itself; this is shown in Figure 4. There appears to be a clear difference in the volatility of the series before and after 1950. If we ignore the spike near 2008, likely due to the financial crisis, the difference in the volatility is more marked still. What is asked in problem 1 (d) is to divide our sample into two parts, 1900-1950 and 1951-2018, and repeat analysis for each subsample separately. As can be expected based on Figure 4, results will differ substantially, at least with respect to the presence of heteroskedasticity.

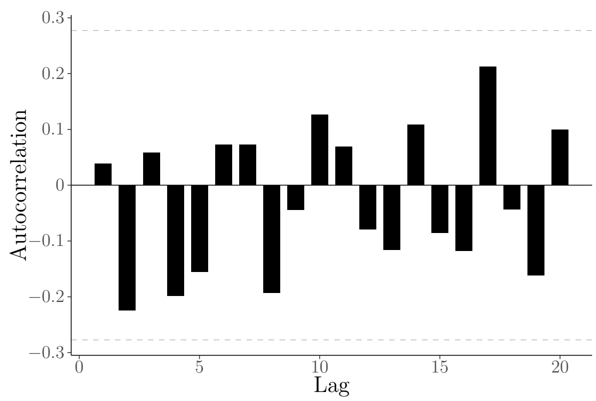
The ACF and PACF for `dlogrdiv` and `dlogrsp` in the subsample 1900-1950 can be seen in Figure 6. The lack of significant lags in the ACF and PACF of either series suggests



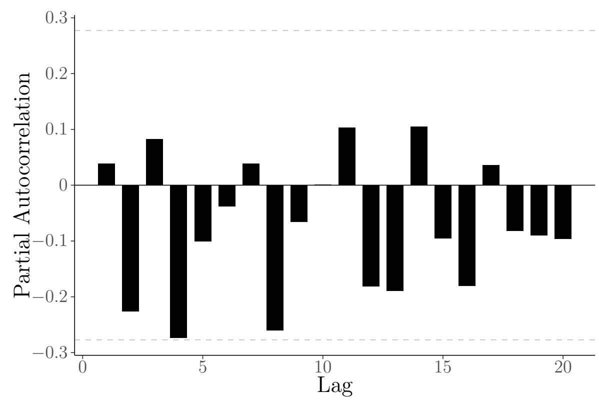
(A) ACF of `dlogrdiv` (1900-1950)



(B) PACF of `dlogrdiv` (1900-1950)



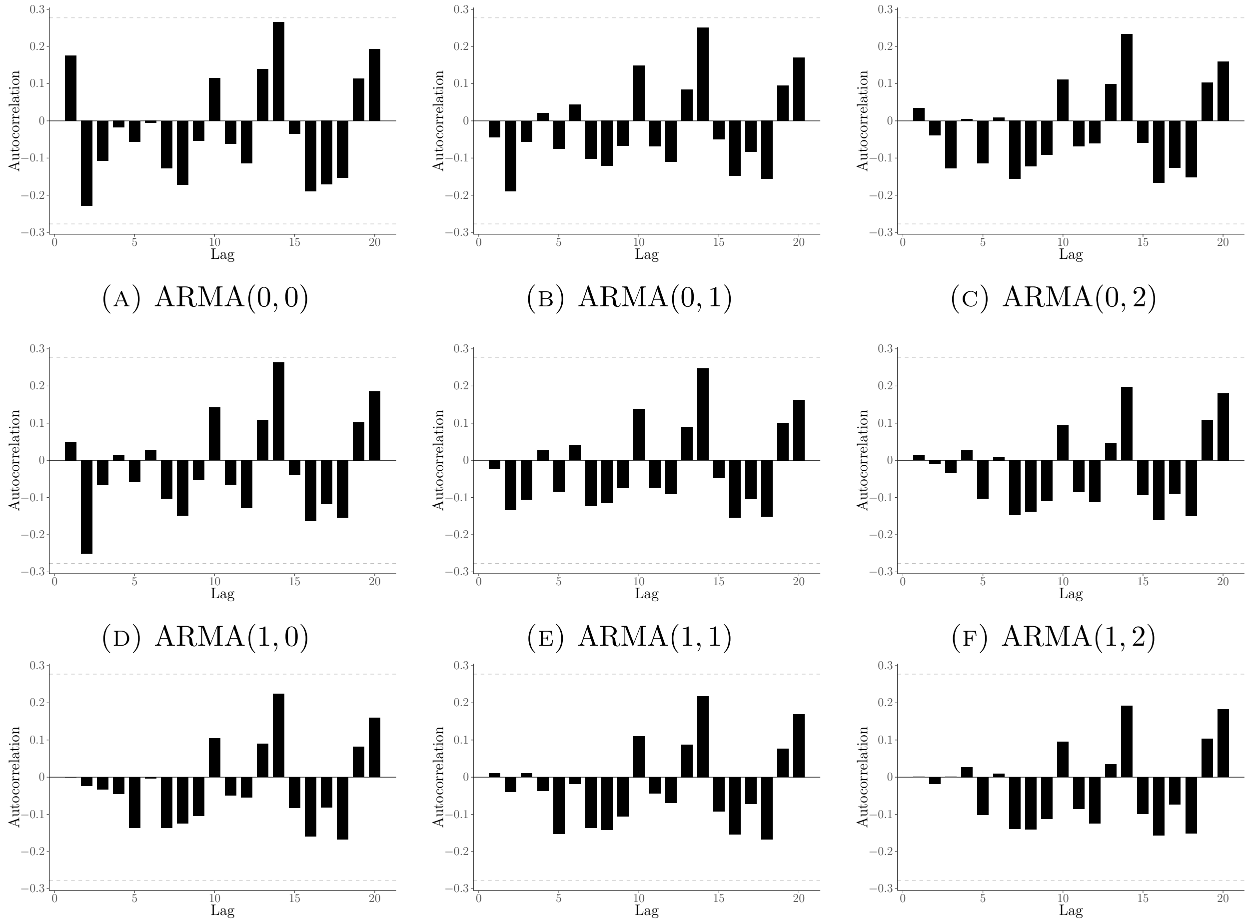
(C) ACF of `dlogrsp` (1900-1950)



(D) PACF of `dlogrsp` (1900-1950)

FIGURE 6. Sample autocorrelation and partial autocorrelation functions for real dividends growth and real stock prices growth in the subsample 1900-1950

that we may not be able to reject the hypothesis that the series are white noise. We will approach this from two angles: in the first place, we will estimate an array of ARMA(p, q) and check the AIC and BIC criteria; if these favor the ARMA$(0, 0)$ model, it is evidence in favor of white noise. We will also test the null hypothesis of the series being white noise, by way of the test proposed by Ljung and Box (1978). As mentioned, an array of ARMA(p, q) models are estimated for `dlogrdiv` and `dlogrsp` in the subsample 1900-1950. The sample ACF of residuals for `dlogrdiv` can be seen in Figure 7. The sample ACF of residuals for `dlogrsp` can be seen in Figure 8. The AIC criteria for the ARMA(p, q) models estimated for `dlogrdiv` in the subsample 1900-1950 can be seen in Table 12. The BIC criteria for the same models can be seen in Table 13. The BIC, which penalizes complexity more heavily than IC, clearly suggests that the ARMA$(0, 0)$ model is the best model for `dlogrdiv` in the subsample 1900-1950, confirming our suspicion of white noise. The AIC does not suggest this model as we might have hoped, instead preferring the ARMA$(1, 2)$ model. Nonetheless,



(A) ARMA$(0,0)$ (B) ARMA$(0,1)$ (C) ARMA$(0,2)$

(D) ARMA$(1,0)$ (E) ARMA$(1,1)$ (F) ARMA$(1,2)$

(G) ARMA$(2,0)$ (H) ARMA$(2,1)$ (I) ARMA$(2,2)$

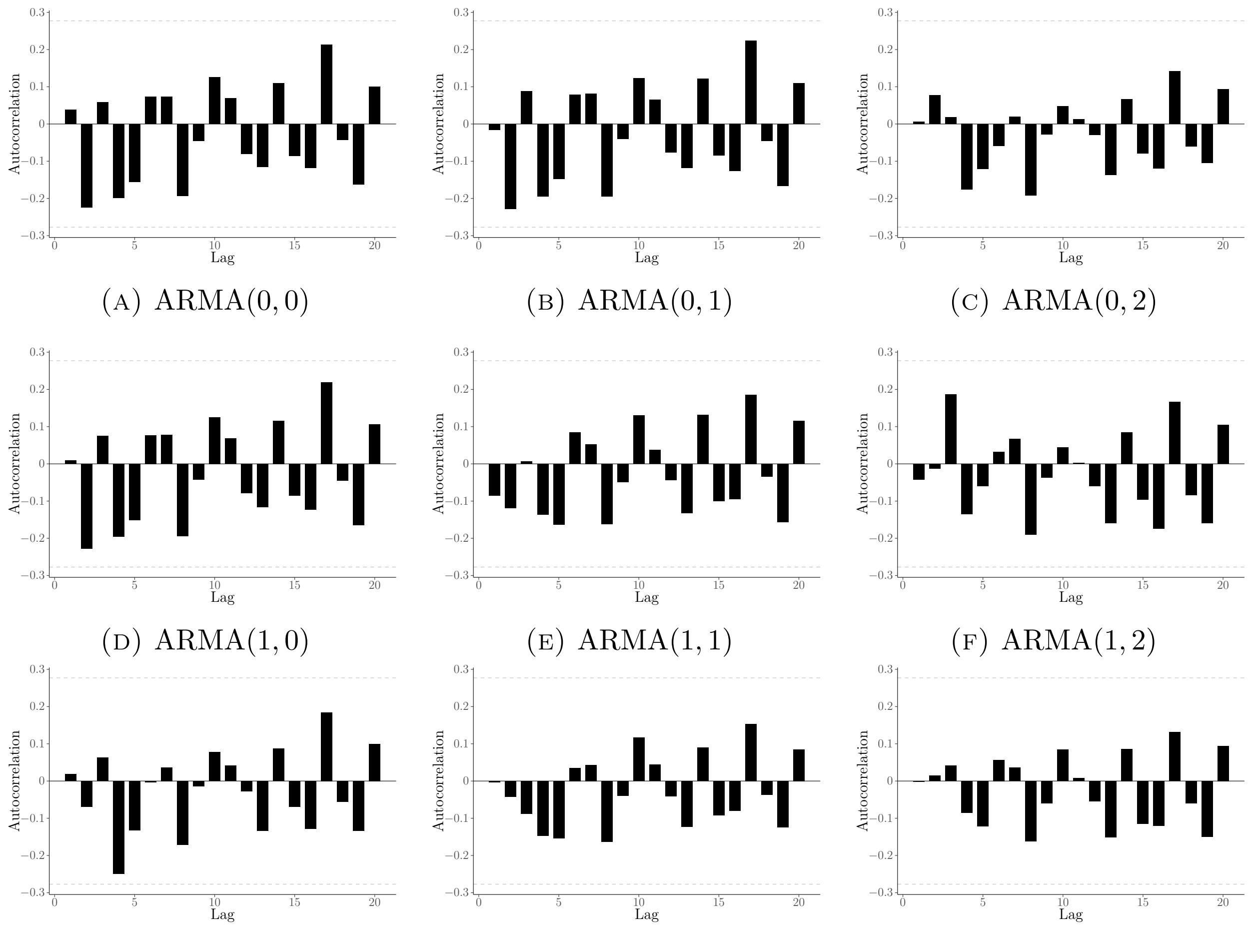
FIGURE 7. Sample autocorrelation functions of residuals for ARMA(p,q) models estimated for `dlogrdiv` (real dividends growth) in the subsample 1900-1950

TABLE 12. AIC values for ARMA(p,q) models estimated for `dlogrdiv` (real dividends growth) in the subsample 1900-1950

	$q=0$	$q=1$	$q=2$
$p=0$	-40.46	-41.33	-41.03
$p=1$	-40.05	-40.04	-45.15
$p=2$	-41.72	-39.77	-43.27

we thus far have relatively weak evidence that the series is not white noise, especially if we note the "clean" residuals in Figure 7a.

The AIC criteria for the ARMA(p,q) models estimated for `dlogrsp` in the subsample 1900-1950 can be seen in Table 14. The BIC criteria for the same models can be seen in Table 15. Again, AIC does not suggest the ARMA$(0,0)$ as would have been convenient, but



(A) ARMA(0, 0) (B) ARMA(0, 1) (C) ARMA(0, 2)

(D) ARMA(1, 0) (E) ARMA(1, 1) (F) ARMA(1, 2)

(G) ARMA(2, 0) (H) ARMA(2, 1) (I) ARMA(2, 2)

FIGURE 8. Sample autocorrelation functions of residuals for ARMA(p, q) models estimated for `dlogrsp` (real stock prices growth) in the subsample 1900-1950

TABLE 13. BIC values for ARMA(p, q) models estimated for `dlogrdiv` (real dividends growth) in the subsample 1900-1950

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-36.64	-35.59	-33.38
$p = 1$	-34.32	-32.39	-35.59
$p = 2$	-34.08	-30.21	-31.80

BIC clearly does. For the same reasons as in the case of `dlogrdiv`, we have evidence in favor of the series being white noise.

We now turn to our second approach, which is to carry out the Ljung-Box test for white noise for each series. The results of the test for `dlogrdiv` in the subsample 1900-1950 can be seen in Table 16. The results of the Ljung-Box test for `dlogrsp` in the subsample 1900-1950

TABLE 14. AIC values for ARMA(p, q) models estimated for `dlogrsp` (real stock prices growth) in the subsample 1900-1950

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-7.07	-5.20	-8.17
$p = 1$	-5.14	-6.83	-8.92
$p = 2$	-5.70	-5.97	-9.28

TABLE 15. BIC values for ARMA(p, q) models estimated for `dlogrsp` (real stock prices growth) in the subsample 1900-1950

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-3.24	0.53	-0.52
$p = 1$	0.60	0.82	0.64
$p = 2$	1.95	3.59	2.20

TABLE 16. Ljung-Box test for white noise for `dlogrdiv` (real dividends growth) in the subsample 1900-1950

Lags	χ^2	p-value
5	5.3028	0.3800
10	9.1522	0.5177
15	16.8376	0.3287
20	28.0538	0.1081

TABLE 17. Ljung-Box test for white noise for `dlogrsp` (real stock prices growth) in the subsample 1900-1950

Lags	χ^2	p-value
5	6.6200	0.2505
10	10.7313	0.3788
15	13.8278	0.5386
20	21.6891	0.3576

can be seen in Table 17. The result of non-rejection of the null hypothesis at typical critical values for each specification strengthens our previous suspicions that the series `dlogrdiv` and `dlogrsp` are accurately modeled as white noise (although potentially with nonzero mean) in the subsample 1900-1950, when considering possible ARMA(p, q) representations.

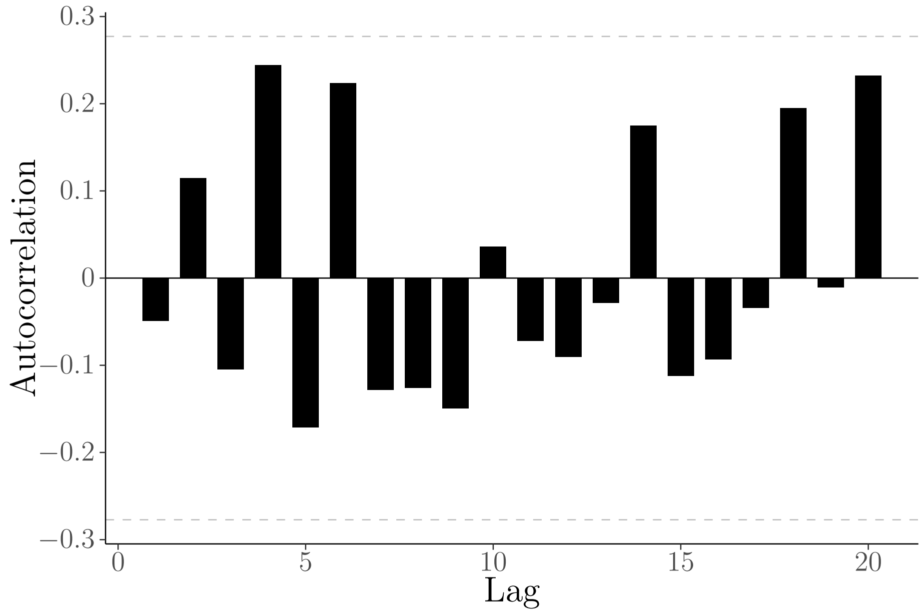


FIGURE 9. Sample autocorrelation function of squared residuals for ARMA$(0, 0)$ model of `dlogrdiv` (real dividends growth) in the subsample 1900-1950

Therefore whether one takes the Box-Jenkins approach of examining residual autocorrelations or decides based upon the BIC, we find that a parsimonious model both for `dlogrdiv` and `dlogrsp` in the subsample 1900-1950 is the ARMA$(0, 0)$ model, which is equivalent to white noise plus a constant term.

We now look for the presence of ARCH effects by plotting the autocorrelation function of the squared residuals (squared deviations from sample mean) of each series, and perform the LM test for ARCH effects. The sample ACF of squared residuals for `dlogrdiv` can be seen in Figure 9. The sample ACF of squared residuals for `dlogrsp` can be seen in Figure 10. The results of the LM test for ARCH effects for `dlogrdiv` in the subsample 1900-1950 can be seen in Table 18. The results of the LM test for ARCH effects for `dlogrsp` in the subsample 1900-1950 can be seen in Table 19. Both the results of the LM tests and the absence of significant autocorrelations in the squared residuals suggest that there is no evidence of ARCH effects in either series in the subsample 1900-1950. This stands in contrast with the results of problem 1 (b), where we found evidence of ARCH effects in `dlogrdiv` in the full sample. This finding is not atypical. As pointed out in the lecture notes of Solá (2025a, p. 10), GARCH effects are said not to "aggregate temporally," meaning that a series that exhibits GARCH effects at a given frequency will not exhibit them at different frequencies.

Summing up, the answer to problem (a) adapted to the subsample 1900-1950 is that we identify both the series `dlogrdiv` and `dlogrsp` as white noise, possibly with a constant term,[3] and no evidence of ARCH effects. The absence of ARCH effects means that solutions to problems (b) and (c) are not applicable in this subsample.

We now turn to the analysis of the second subsample, 1951-2018. The ACF and PACF for `dlogrdiv` and `dlogrsp` in the subsample 1951-2018 can be seen in Figure 11. We can

[3]Sample means are not significantly different from zero and are thus not reported.

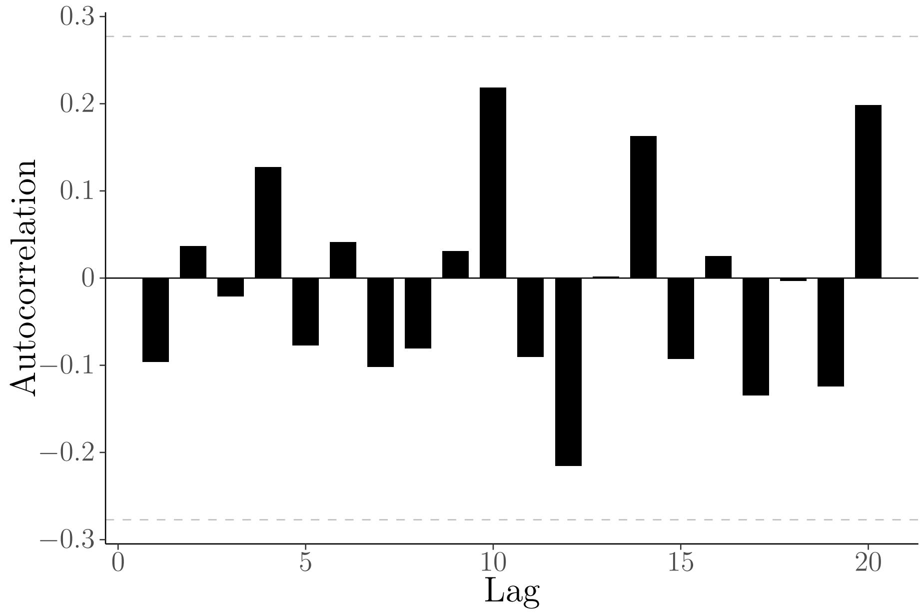


FIGURE 10. Sample autocorrelation function of squared residuals for ARMA$(0, 0)$ model of `dlogrsp` (real stock prices growth) in the subsample 1900-1950

observe significant early lags in `dlogrdiv` which are suggestive for ARMA modeling, and a significant lag at 19 for `dlogrsp`, which is not likely to be relevant. As in previous analysis, we estimate an array of ARMA(p, q) models and check the AIC and BIC criteria to make a well-informed decision on the best model. The sample ACF of residuals for the various models of `dlogrdiv` in the subsample 1951-2018 can be seen in Figure 12. Every plot except for 12a which is by construction identical to the original 11a exhibits "clean" residuals.

The sample ACF of residuals for the various models of `dlogrsp` in the subsample 1951-2018 can be seen in Figure 13. Inspection of Figure 13 puts us in a similar position as in the analysis of the previous subsample. Analysis of AIC and BIC and the results of the ARCH test will put us in a better position to decide on the model.

For the array of ARMA(p, q) models estimated for `dlogrdiv` in the subsample 1951-2018, the AIC can be seen in Table 20. The BIC for the same models can be seen in Table 21. The AIC are more or less similar for all models except for the ARMA$(0, 0)$, which is in line with the result of a visual inspection of the residual ACFs in Figure 12. The favored model according to this criterion is ARMA$(2, 0)$ = AR(2), as in the full-sample analysis. According to the BIC, the preferred model is ARMA$(0, 1)$ = MA(1); The ARMA$(2, 0)$ is a "close" second. We might favor the ARMA$(0, 1)$, which is third-best according to AIC, in the name of parsimony. However, in the full-sample analysis we modeled the series with an AR(2) process and for this reason this will be the model chosen (justified also, of course, by the fact that it is the AIC choice).

The AIC for the array of ARMA(p, q) models estimated for `dlogrsp` in the subsample 1951-2018 can be seen in Table 22. The BIC for the same models can be seen in Table 23.

As happened in the prior analysis, BIC clearly prefers the white noise model while AIC places it in second place, after ARMA$(2, 2)$. Supplementing the approach with the results of

Table 18. LM test for ARCH effects for `dlogrdiv` (real dividends growth) in the subsample 1900-1950

Lags	χ^2	p-value
1	0.1172	0.7320
2	0.7608	0.6836
3	1.1551	0.7638
4	3.5294	0.4734
5	4.3200	0.5043
6	5.6440	0.4642
7	5.7129	0.5736
8	8.1680	0.4172
9	8.0439	0.5297
10	8.3201	0.5976
11	8.2151	0.6939
12	8.6080	0.7360
13	8.5764	0.8041
14	13.2910	0.5038
15	14.5835	0.4818
16	14.2451	0.5805
17	14.3563	0.6417
18	16.9160	0.5289
19	16.6034	0.6167
20	21.6816	0.3580

the Ljung-Box white noise test in Table 24, we choose to model `dlogrsp` as an ARMA$(0, 0)$ process.

Now, assuming the ARMA$(2, 0)$ model for `dlogrdiv` and ARMA$(0, 0)$ for `dlogrsp`, we test for the presence of ARCH effects using the Engle test, the results of which can be seen in Tables 25 and 26. It is safe to say that we do not detect the presence of ARCH effects in either series in the subsample 1951-2018.[4] Again, this is in line with the fact that GARCH effects do not aggregate temporally. Summing up, the answer to problem (a) adapted to the subsample 1951-2018 is that we identify `dlogrdiv` as an AR(2) process and `dlogrsp` as white noise, possibly with a constant term, and no evidence of ARCH effects. Problems (b) and (c), in light of the absence of ARCH effects, are not applicable in this subsample either.

[4]We might think to estimate an ARCH model of order 1, basically to allow the volatility to adjust for the spike in volatility near 2008 (see Figure 4), but it could be argued that to do so would be overfitting. Personally, I do not think the leap to a GARCH model is worth the added complexity for this particular model for the subsample.

TABLE 19. LM test for ARCH effects for `dlogrsp` (real stock prices growth) in the subsample 1900-1950

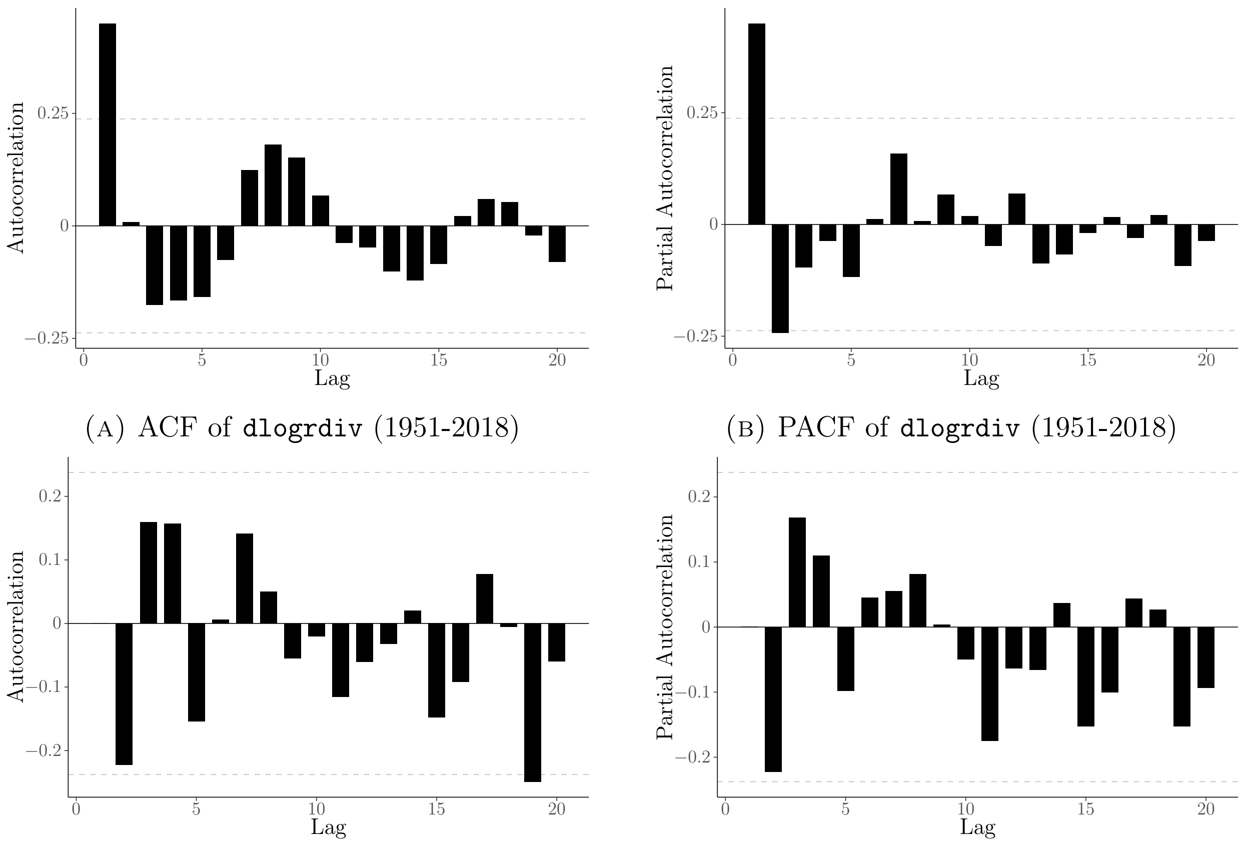
Lags	χ^2	p-value
1	0.4612	0.4971
2	0.5764	0.7496
3	0.6409	0.8870
4	1.3523	0.8524
5	1.4159	0.9226
6	1.4446	0.9631
7	1.7433	0.9727
8	2.6197	0.9559
9	2.7414	0.9737
10	5.5967	0.8479
11	5.9277	0.8781
12	9.1239	0.6923
13	9.7451	0.7146
14	11.3460	0.6587
15	11.0422	0.7496
16	10.8834	0.8166
17	12.3885	0.7761
18	11.9846	0.8480
19	13.4972	0.8122
20	18.3112	0.5669

TABLE 20. AIC values for ARMA(p, q) models estimated for `dlogrdiv` (real dividends growth) in the subsample 1951-2018

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-186.65	-202.99	-203.16
$p = 1$	-201.10	-202.27	-201.49
$p = 2$	-203.91	-202.66	-200.68

TABLE 21. BIC values for ARMA(p, q) models estimated for `dlogrdiv` (real dividends growth) in the subsample 1951-2018

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-182.22	-196.33	-194.28
$p = 1$	-194.45	-193.39	-190.40
$p = 2$	-195.04	-191.56	-187.36



(A) ACF of `dlogrdiv` (1951-2018)

(B) PACF of `dlogrdiv` (1951-2018)

(C) ACF of `dlogrsp` (1951-2018)

(D) PACF of `dlogrsp` (1951-2018)

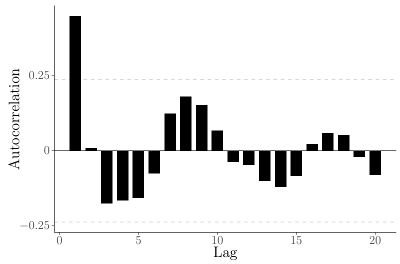
FIGURE 11. Sample autocorrelation and partial autocorrelation functions for real dividends growth and real stock prices growth in the subsample 1951-2018

TABLE 22. AIC values for ARMA(p, q) models estimated for `dlogrsp` (real stock prices growth) in the subsample 1951-2018

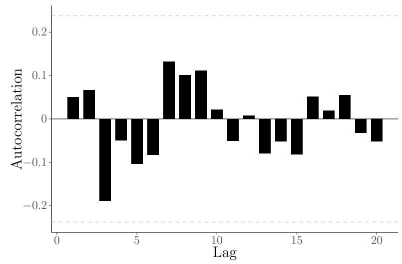
	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-46.95	-44.95	-45.79
$p = 1$	-44.95	-44.91	-44.03
$p = 2$	-46.35	-45.19	-49.82

TABLE 23. BIC values for ARMA(p, q) models estimated for `dlogrsp` (real stock prices growth) in the subsample 1951-2018

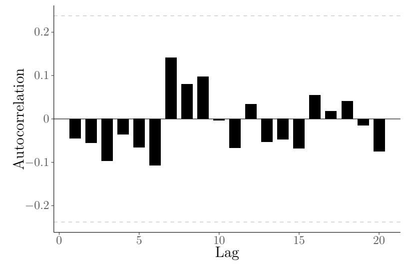
	$q = 0$	$q = 1$	$q = 2$
$p = 0$	-42.51	-38.29	-36.92
$p = 1$	-38.29	-36.03	-32.93
$p = 2$	-37.48	-34.10	-36.51



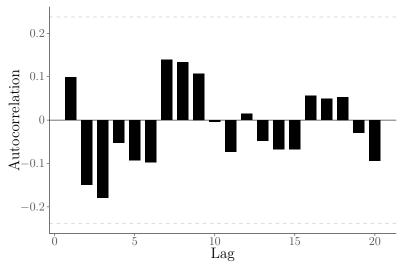
(A) ARMA(0, 0)



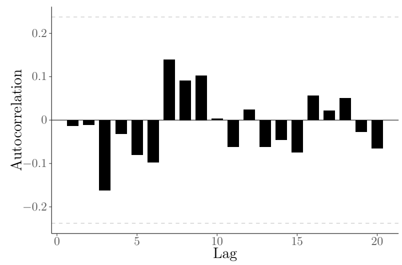
(B) ARMA(0, 1)



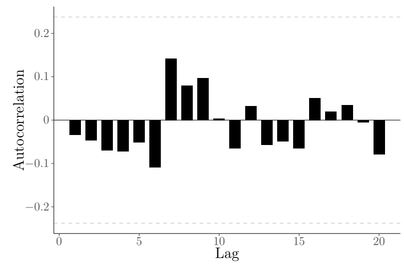
(C) ARMA(0, 2)



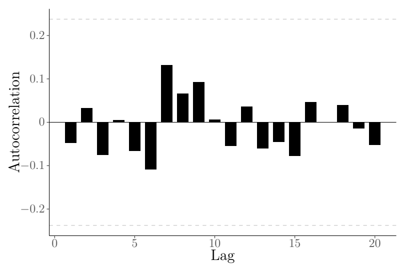
(D) ARMA(1, 0)



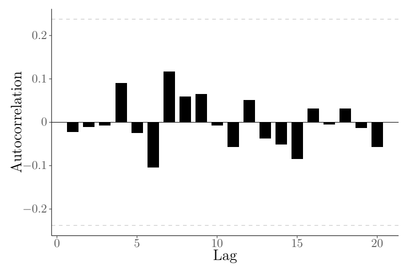
(E) ARMA(1, 1)



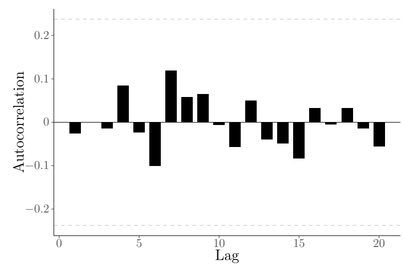
(F) ARMA(1, 2)



(G) ARMA(2, 0)



(H) ARMA(2, 1)

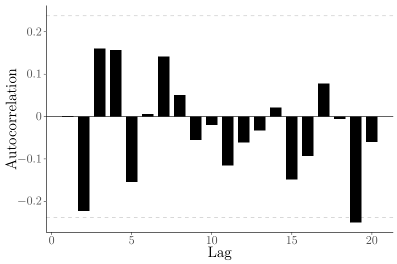


(I) ARMA(2, 2)

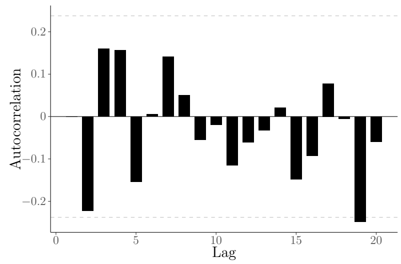
FIGURE 12. Sample autocorrelation functions of residuals for ARMA(p, q) models estimated for `dlogrdiv` (real dividends growth) in the subsample 1951-2018

TABLE 24. Ljung-Box test for white noise for `dlogrsp` (real stock prices growth) in the subsample 1951-2018

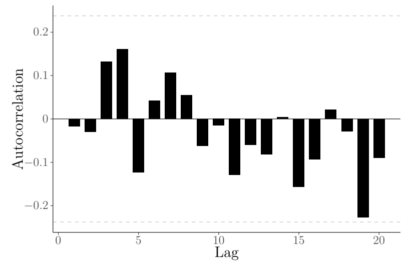
Lags	χ^2	p-value
5	9.0633	0.1066
10	11.0976	0.3500
15	14.6056	0.4802
20	22.3232	0.3232



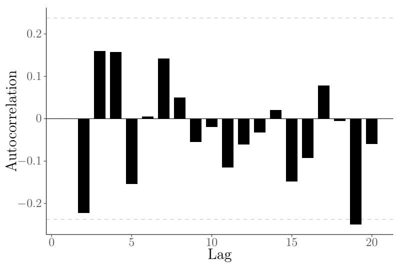
(A) ARMA$(0,0)$



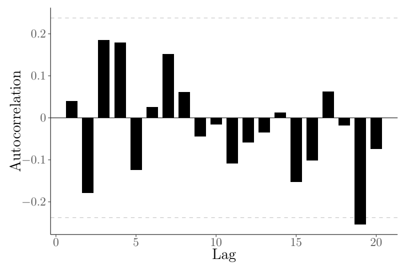
(B) ARMA$(0,1)$



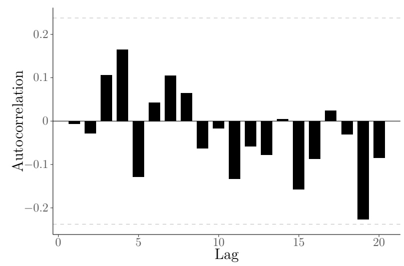
(C) ARMA$(0,2)$



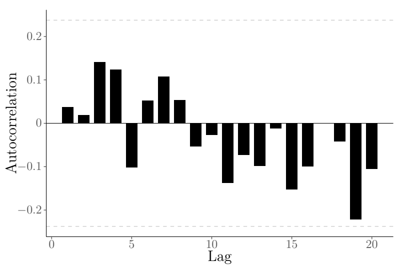
(D) ARMA$(1,0)$



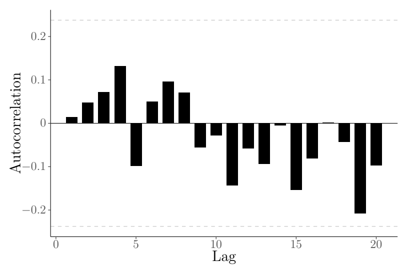
(E) ARMA$(1,1)$



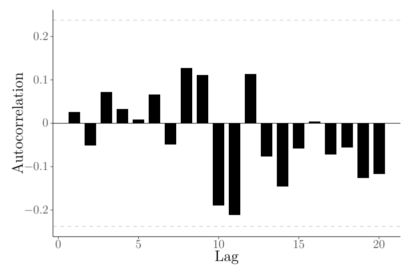
(F) ARMA$(1,2)$



(G) ARMA$(2,0)$



(H) ARMA$(2,1)$



(I) ARMA$(2,2)$

FIGURE 13. Sample autocorrelation functions of residuals for ARMA(p,q) models estimated for `dlogrsp` (real stock prices growth) in the subsample 1951-2018

Table 25. LM test for ARCH effects for `dlogrdiv` (real dividends growth) in the subsample 1951-2018

Lags	χ^2	p-value
1	3.9401	0.0471
2	4.3625	0.1129
3	4.3542	0.2257
4	4.2851	0.3688
5	4.3660	0.4980
6	4.4725	0.6130
7	4.3973	0.7330
8	4.3359	0.8256
9	4.3545	0.8866
10	4.4448	0.9251
11	4.5719	0.9501
12	4.8896	0.9616
13	4.9990	0.9752
14	5.6411	0.9747
15	6.2203	0.9758
16	6.4537	0.9824
17	6.8810	0.9850
18	6.9020	0.9909
19	7.0589	0.9939
20	7.0819	0.9964

Table 26. LM test for ARCH effects for `dlogrsp` (real stock prices growth) in the subsample 1951-2018

Lags	χ^2	p-value
1	0.7780	0.3777
2	2.5636	0.2775
3	2.7581	0.4304
4	2.8472	0.5837
5	2.6720	0.7504
6	2.5509	0.8627
7	2.6264	0.9173
8	2.7091	0.9513
9	3.8294	0.9223
10	4.7301	0.9085
11	4.6720	0.9460
12	4.6310	0.9692
13	4.5360	0.9840
14	6.3448	0.9570
15	6.6167	0.9674
16	6.5816	0.9805
17	7.1374	0.9817
18	8.0549	0.9778
19	7.8678	0.9880
20	8.2169	0.9903

2. Prices and Money

Problem 2. Use the data for USA included in the file `datos_new.wf1`. Consider the following variables: $\log(m_t)$, $\log(p_t)$, $\log(y_t)$ and i_t, which represent the log of money (`m1`), prices (`cpiindex`), output (`gdp`) and an interest rate (`intrates`). For the period 1962q1-2014q4:

(a) Determine the order of integration of each variable.
(b) Determine how many cointegration vectors you find.
(c) Estimate the following equation

$$\log(m_t) - \log(p_t) = \alpha_0 + \alpha_1 i_t + \alpha_2 \log(y_t) + \varepsilon_t \tag{5}$$

Report your results. Can you interpret this equation as a short or long-run relationship?

(d) Repeat (c) for the following equations:

$$\Delta \log(m_t) - \Delta \log(p_t) = \alpha_0 + \alpha_1 i_t + \alpha_2 \Delta \log(y_t) + \varepsilon_t \tag{6}$$

and

$$\Delta \log(m_t) - \Delta \log(p_t) = \alpha_0 + \alpha_1 \Delta i_t + \alpha_2 \Delta \log(y_t) + \varepsilon_t \tag{7}$$

Critically comment on the similarities and differences between the results of (c) and the regressions in (d). How would you modify the demand for money if you expect an Error Correction Model representation to be valid?

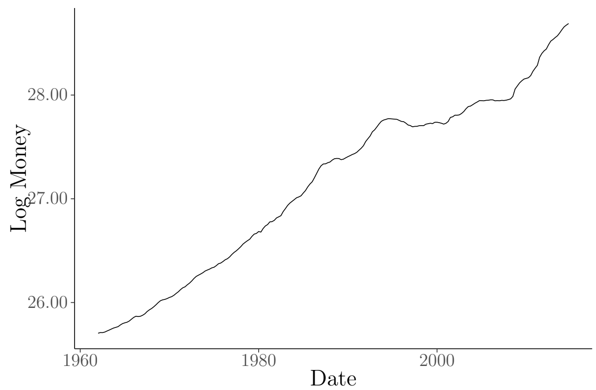
Solution 2 (a). The order of integration of a series is, oversimplifying slightly, the number of consecutive differences that must be taken to obtain a stationary series. The variables `logm`, `logp`, `logy` and `i` are respectively the log of the money, the log of the price level, the log of output, and the interest rate. Money, prices, and output typically grow in a roughly exponential fashion, which motivates the decision to take logs, and the fact that there is growth will suggest non-stationarity or integration of order one. Money, prices, and interest rates are susceptible to policy interventions and tend to exhibit markedly different behavior in the different macroeconomic contexts encompassed by the sample: another reason to expect non-stationarity. Plots of each series in logs and the interest rate can be seen in Figure 14.

In order to determine the order of integration of each variable, we will use the Augmented Dickey-Fuller (ADF) test (Dickey and Fuller, 1979) described as follows in Pfaff (2008, p. 61). Let y_t be the series to be tested. First, a number of lags k to use in the test must be chosen. There are several methods to select the number of number of lags, including but not limited to the AIC and the "*general-to-specific* method." The *general-to-specific* method seems to be preferred, as noted in Solá (2025b, p. 22) and Pfaff (2008, p. 60), but we will use AIC here for simplicity and personal preference. Given k, the ADF test equation

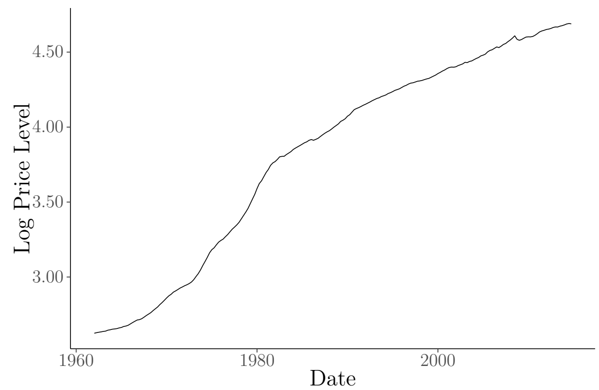
$$\Delta y_t = \beta_1 + \beta_2 t + \pi y_{t-1} + \sum_{j=1}^{k} \gamma_j \Delta y_{t-j} + u_t \tag{8}$$

is estimated. The procedure is then described as follows:

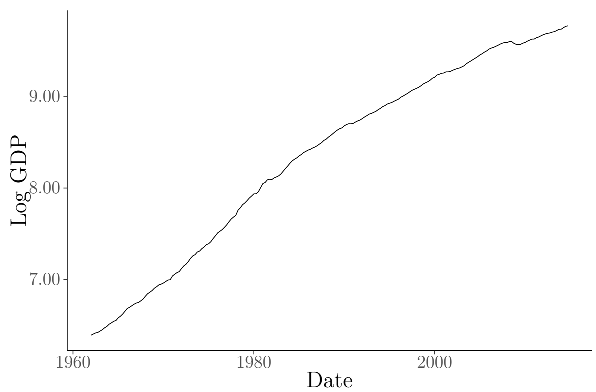
> [...] the testing strategy starts by testing if $\pi = 0$ using the t statistic τ_τ. This statistic is not standard Student t distributed, but critical values can be found in Fuller (1976). If this test is rejected, then there is no need to proceed further. The testing sequence is continued by an F type test Φ_3



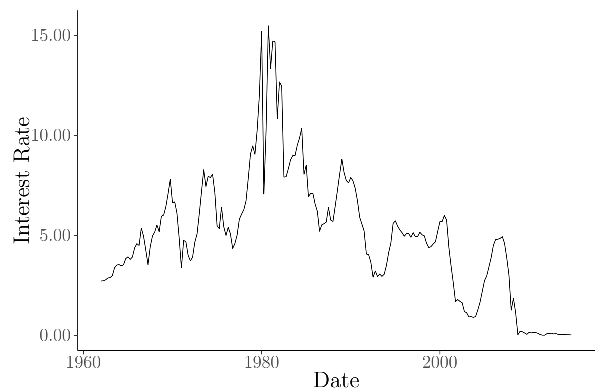
(A) Log of money (`logm`)



(B) Log of prices (`logp`)



(C) Log of output (`logy`)



(D) Interest rate (`i`)

FIGURE 14. Plots of log of money, log of prices, log of output, and interest rate

> with $H_0 : \beta_2 = \pi = 0$ using the critical values tabulated in Dickey and Fuller (1981). If it is significant, then test again for a unit root using the standardized normal. Otherwise, if the hypothesis $\beta_2 = 0$ cannot be rejected, reestimate Equation (8) but without a trend. The corresponding t and F statistics for testing if $H_0 : \pi = 0$ and $H_0 : \beta_1 = \pi = 0$ are denoted by $\tau_\mu(\tau)$ and Φ_1. Again, the critical values for these test statistics are provided in the literature cited above. If the null hypothesis of $\tau_\mu(\tau)$ is rejected, then there is again no need to go further. If it is not, then employ the F statistic Φ_1 for testing the presence of a constant and a unit root. (Pfaff, 2008, p. 60)

We follow the procedure outlined above, using AIC for lag selection.[5] We begin by estimating the full equation and testing the null of $\pi = 0$. The results of the ADF test for each variable are shown in Table 27. The statistic of interest for testing the aforementioned null is named τ_τ in the output, and as we can see upon comparison with the critical values, we

[5]There is a step in the procedure that I do not follow, the step that follows the rejection of the F type test immediately after failing to reject the null in the first step. The author refers to some sort of standardized normal test with which I am not familiar. In any case, the final results that are derived after skipping this step are credible based on economic theory and the cointegration analysis that follows.

TABLE 27. Augmented Dickey-Fuller (trend specification) test results

Variable	τ_τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
$\log(m_t)$	-1.408	-3.99	-3.43	-3.13	1
$\log(p_t)$	0.189	-3.99	-3.43	-3.13	1
$\log(y_t)$	1.094	-3.99	-3.43	-3.13	1
i_t	-2.692	-3.99	-3.43	-3.13	1

fail to reject at even the 10% level. The next is to remove the trend term (impose $\beta_2 = 0$) and test the null again. The results of the ADF test without the trend term are shown in Table 28. We fail to reject at the 10% level for all variables except for $\log(y_t)$, for which

TABLE 28. Augmented Dickey-Fuller (`drift` specification) test results

Variable	τ_μ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
$\log(m_t)$	0.113	-3.46	-2.88	-2.57	1
$\log(p_t)$	-2.235	-3.46	-2.88	-2.57	1
$\log(y_t)$	-4.712	-3.46	-2.88	-2.57	1
i_t	-1.869	-3.46	-2.88	-2.57	1

we reject at the 1% level. This is challenging; economic theory would strongly suggest that this variable should be integrated of order one. It does make sense though, given the plot in Figure 14c, that incorporating a drift term, that is a nonzero mean in the growth rate, would lead to a more or less stationary series. Intuitively, my interpretation of the problem is that if there were a unit root, there were not innovations in the series large enough to suggest that the persistent increase in the series is due to persistence of the innovation rather than a deterministic constant in the growth rate. The decision of how to resolve this dilemma is not clear-cut, but we will proceed with the modeling choice of treating the empirical trend as due to a unit root rather than a deterministic trend.[6]

For the other variables, we properly conclude the procedure by carrying out the ADF test with $\beta_1 = 0$ and $\beta_2 = 0$; that is, with neither a trend nor a drift term. The results of the ADF test for this specification are shown in Table 29. We now fail to reject the null for all variables, including $\log(y_t)$, as we might have expected, though we have abandoned the procedure, strictly speaking, for this variable.

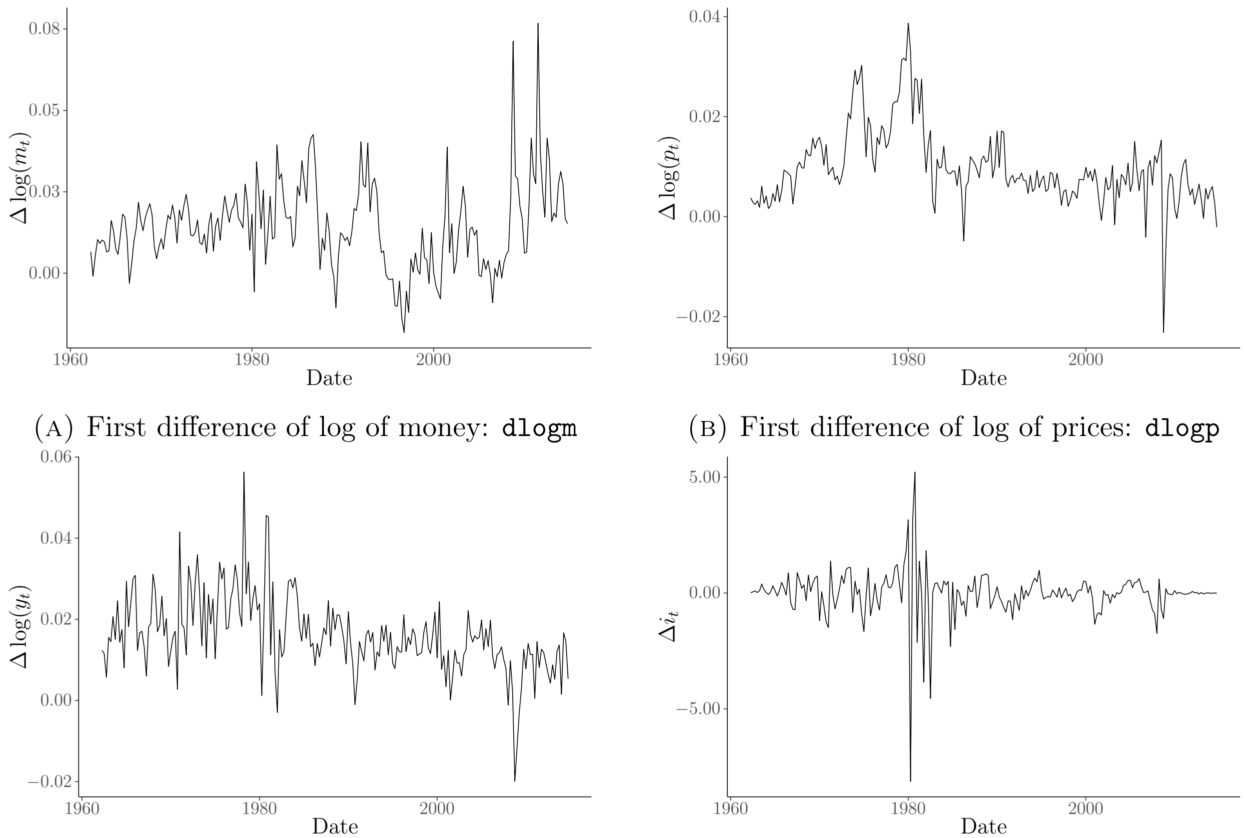
We therefore conclude that the series are non-stationary and terminate the procedure.

Thus far, we have rejected that the series are $I(0)$, with a caveat for $\log(y_t)$. We now test whether their differenced series, `dlogm`, `dlogp`, `dlogy` and `di`, are $I(0)$. The differenced series are plotted in Figure 15. That is, we apply the same procedure as above, but now to

[6]C. R. Nelson and Plosser (1982) is concerned with precisely the issue at hand. From the abstract: "This paper investigates whether macroeconomic time series are better characterized as stationary fluctuations around a deterministic trend or as non-stationary processes that have no tendency to return to a deterministic path. Using long historical time series for the U.S. we are unable to reject the hypothesis that these series are non-stationary stochastic processes with no tendency to return to a trend line."

TABLE 29. Augmented Dickey-Fuller (`none` specification) test results

Variable	τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
$\log(m_t)$	5.036	-2.58	-1.95	-1.62	1
$\log(p_t)$	3.366	-2.58	-1.95	-1.62	1
$\log(y_t)$	6.185	-2.58	-1.95	-1.62	1
i_t	-1.157	-2.58	-1.95	-1.62	1



(A) First difference of log of money: `dlogm`

(B) First difference of log of prices: `dlogp`

(C) First difference of log of output: `dlogy`

(D) First difference of interest rate: `di`

FIGURE 15. Plots of first differences of log of money, log of prices, log of output, and interest rate

the first differences of the series. The results of the ADF test for the first differences of each variable are shown in Table 30. As we can see, we now reject the null at (significantly less than) the 1% level for all differenced series. We can conclude that the series are integrated of order one ($I(1)$).

Solution 2 (b). We now turn to the analysis of cointegration, which asks whether there are linear combinations of the variables that are stationary. The method to be used is the trace test of Johansen (1991), which seeks to find the number of cointegration vectors r based on the results of a sequence of hypothesis tests involving the trace of a matrix. The procedure

TABLE 30. Augmented Dickey-Fuller (`trend` specification) test results for first differences

Variable	τ_τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
$\Delta \log(m_t)$	-5.414	-3.99	-3.43	-3.13	1
$\Delta \log(p_t)$	-4.714	-3.99	-3.43	-3.13	1
$\Delta \log(y_t)$	-7.186	-3.99	-3.43	-3.13	1
Δi_t	-13.963	-3.99	-3.43	-3.13	1

is described in detail in Pfaff (2008, p. 81). We select the number of lags k to use in the underlying VAR based on the AIC, which gives $k = 2$. The results of the trace test are shown in Table 31.

TABLE 31. Johansen Trace Test Results

Hypothesis	Test Statistic	Critical Values 10%	5%	1%
$r \leq 3$	3.456	7.52	9.24	12.97
$r \leq 2$	8.012	17.85	19.96	24.60
$r \leq 1$	47.896	32.00	34.91	41.07
$r = 0$	124.574	49.65	53.12	60.16

The results in Table 31 show that we reject the null hypothesis of no cointegration ($r = 0$) and of at most one cointegrating vector ($r = 1$) at the 1% level. However, we fail to reject $r \leq 2$ and $r \leq 3$ at even the 10% level. The method followed thus stipulates that we may conclude that there may be two cointegration vectors.

Solution 2 (c). The results of the OLS regression of Equation (5) are shown in Table 32. As we can see, the linear regression fits the data well and all coefficients are significant. The signs also make sense if we think of $\log m_t - \log p_t$ as real money demand, or real money balances: output increases demand and interest rates decrease it. Now, this might be interpreted as a long-run relationship; the results of the Johansen procedure suggested that there are cointegration vectors. We will now see, however, that this is not the case.

We follow the procedure outlined above to test whether the residuals of the regression are stationary, following the approach of Engle and Granger (1987). If they are, then we can interpret the relationship as an instance of the cointegration vector. If we find that the residuals are not $I(0)$, this interpretation fails and we may therefore call the relationship a short-run relationship. The first step in the procedure is to test the unrestricted regression of Equation (8) using the residuals of the regression in Equation (5) as the dependent variable. The results of the ADF test for the residuals are shown in Table 33. The first row corresponds to the first version of the test. We clearly fail to reject the null. What the procedure prescribes next is to test the null in the drift specification; this is the second row of Table 33. Again, we fail to reject the null. The next and final step before concluding that the there is a unit root in the residuals is to test the null under the specification without trend nor drift. Again, as can be seen in the third row of Table 33, we fail to reject the null. Therefore we

Table 32. OLS Regression Results for Equation (5)

	Dependent variable:
	$\log(m_t) - \log(p_t)$
i_t	-0.029^{***}
	(0.002)
$\log(y_t)$	0.120^{***}
	(0.007)
Constant	22.475^{***}
	(0.065)
Observations	212
R^2	0.765
Adjusted R^2	0.763
Residual Std. Error	0.101 (df = 209)
F Statistic	340.050^{***} (df = 2; 209)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 33. Augmented Dickey-Fuller Test on Residuals of Equation (5)

Test Specification	Test Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Trend	−1.039	−3.99	−3.43	−3.13	1
Drift	−1.053	−3.46	−2.88	−2.57	1
None	−1.080	−2.58	−1.95	−1.62	1

conclude that the residuals are non-stationary and that the relationship in Equation (5) is a short-run relationship.

Solution 2 (d). We now estimate Equations (6) and (7), restated below for convenience.

$$\Delta \log(m_t) - \Delta \log(p_t) = \alpha_0 + \alpha_1 i_t + \alpha_2 \Delta \log(y_t) + \varepsilon_t \tag{9}$$

$$\Delta \log(m_t) - \Delta \log(p_t) = \alpha_0 + \alpha_1 \Delta i_t + \alpha_2 \Delta \log(y_t) + \varepsilon_t. \tag{10}$$

The results of the OLS regression of Equation (9) are shown in Table 34. The results of the OLS regression of Equation (10) are shown in Table 35. Let us first note that the left hand side of the equations can be interpreted as roughly the percentage change in real money balances, in light of the manipulation

$$\Delta \log(m_t) - \Delta \log(p_t) = \log\left(\frac{m_t}{p_t}\right) - \log\left(\frac{m_{t-1}}{p_{t-1}}\right).$$

We now turn to brief commentary on the results of the regressions, with the obvious disclaimer that since there are endogenous relationships, causal analysis is definitely not

TABLE 34. OLS Regression Results for Equation (9)

	Dependent variable:
	$\Delta \log(m_t) - \Delta \log(p_t)$
i_t	-0.002^{***}
	(0.0003)
$\Delta \log(y_t)$	-0.159
	(0.116)
Constant	0.017^{***}
	(0.002)
Observations	211
R^2	0.195
Adjusted R^2	0.187
Residual Std. Error	0.014 (df = 208)
F Statistic	25.225^{***} (df = 2; 208)
Note:	*p<0.1; **p<0.05; ***p<0.01

TABLE 35. OLS Regression Results for Equation (10)

	Dependent variable:
	$\Delta \log(m_t) - \Delta \log(p_t)$
Δi_t	0.002^{**}
	(0.001)
$\Delta \log(y_t)$	-0.518^{***}
	(0.117)
Constant	0.013^{***}
	(0.002)
Observations	211
R^2	0.089
Adjusted R^2	0.080
Residual Std. Error	0.015 (df = 208)
F Statistic	10.149^{***} (df = 2; 208)
Note:	*p<0.1; **p<0.05; ***p<0.01

appropriate and partial effects should not be though of as necessarily structural in any sense. The positive coefficient on the interest rate in Table 34 suggests that a higher interest rate leads to a less positive change in real money balances. A higher growth in output leads also to less growth in money balances, which is somewhat counterintuitive, but statistically insignificant. The constant is positive, which basically reflects the fact that the real value of money balances has grown over time. Compared to the results of the regression from solution 2 (c), the fit of the regression is much lower, as measured by the R^2 statistics.

In the second regression, Table 35, the interest rate is replaced by its first difference. The coefficient on the change in interest rate is positive, which interpreted literally means that a higher rise in interest rates leads to a lower growth in real money balances. The opposite is true for output growth; higher growth is accompanied by lower growth in real money balances. The constant is again positive, and again the fit is low.

Commentary on estimates aside, we now turn to the more relevant analysis of whether any of these regressions can be thought of as short-run or long-run relationships. The second equation, Equation (9), can be discarded as a long-run relationship *a priori*; we determined in solution 2 (a) that the series are $I(1)$, thus their differences do not cointegrate.

The first relationship is odd. It relates the level of an $I(1)$ variable (the interest rate) to the first differences of $I(1)$ variables. Intuition suggests that this relationship will surely not be a long-run relationship, that is, one for which the residuals are stationary as deviations from a long-run *equilibrium*. To attempt to fit this regression as a cointegration vector would be a gross misapplication of the theory.

I expect an Error Correction Model representation to be valid, in light of the results of the Johansen procedure, which suggested that there are cointegration vectors. I am not sure that my approach is entirely econometrically sound, but it is as follows. We found, crucially, in solution 2 (b) that there is likely a linear combination of the $I(1)$ variables $\log(m_t)$, $\log(p_t)$, $\log(y_t)$ and i_t that is $I(0)$. This linear combination typically normalizes one coefficient. But inspecting Equation (5) for money demand, we have done more than normalize one coefficient, as is made obvious by the manipulation

$$\varepsilon_t = \tilde{\alpha}_0 + \log(m_t) - \log(p_t) + \tilde{\alpha}_1 i_t + \tilde{\alpha}_2 \log(y_t),$$

where $\tilde{\alpha}_0 = -\alpha_0$, $\tilde{\alpha}_1 = -\alpha_1$ and $\tilde{\alpha}_2 = -\alpha_2$. The lack of an independent coefficient on either $\log(m_t)$ or $\log(p_t)$ means that unless the cointegration vector happened to have opposite coefficients on $\log(m_t)$ and $\log(p_t)$, it will necessarily not be described by Equation (5). I therefore suspect that if we remove this restriction, estimating either

$$\log(m_t) - \log(p_t) = \beta_0 + \beta_1 i_t + \beta_2 \log(y_t) + \beta_3 \log(m_t) + \varepsilon_{\beta,t} \tag{11}$$

or

$$\log(m_t) - \log(p_t) = \gamma_0 + \gamma_1 i_t + \gamma_2 \log(y_t) + \gamma_3 \log(p_t) + \varepsilon_{\gamma,t}, \tag{12}$$

we might hope to obtain stationary residuals $\varepsilon_{\beta,t}$ or $\varepsilon_{\gamma,t}$. The results of estimating Equation (11) by OLS are shown in Table 36. Note the extremely high R^2. Following as in solution 2 (c) the procedure of Engle and Granger (1987), we test the residuals of the regression in Equation (11) for stationarity. The results of the ADF test for the residuals, with all three specifications of the test, are shown in Table 37. The first step of the testing procedure of Pfaff (2008) is to test the null of a unit root in the residuals using the full specification of the ADF test, the results displayed in the first row of Table 37. We fail to reject at all levels. The next step is therefore to test the null under the drift specification, which is the second row of Table 37. We now reject at the 5% level, but not at the 1% level. This might

Table 36. OLS Regression Results for Equation (11)

	Dependent variable:
	$\log(m_t) - \log(p_t)$
i_t	-0.012^{***}
	(0.001)
$\log(y_t)$	-0.544^{***}
	(0.024)
$\log(m_t)$	0.852^{***}
	(0.031)
Constant	4.811^{***}
	(0.635)
Observations	212
R^2	0.950
Adjusted R^2	0.950
Residual Std. Error	0.046 (df = 208)
F Statistic	$1{,}325.291^{***}$ (df = 3; 208)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 37. Augmented Dickey-Fuller Test on Residuals of Equation (11)

Test Specification	Test Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Trend	−2.883	−3.99	−3.43	−3.13	1
Drift	−2.897	−3.46	−2.88	−2.57	1
None	−2.903	−2.58	−1.95	−1.62	1

be satisfactory, but suppose it were not; we would then proceed to test the null under the specification without trend nor drift, which is the third row of Table 37. And under this specification, we reject at even the 1% level. We would therefore conclude that the residuals do not have a unit root and that the relationship in Equation (11) is a long-run relationship. Therefore the modification given by Equation (11) is the modification I would propose.

What I cannot account for is the fact that I fail to estimate a cointegration vector by means of OLS on Equation (12). The ADF results on the residuals of this regression are displayed in Table 38, and as can be seen they do not appear to be $I(0)$. I am not sure why this is

TABLE 38. Augmented Dickey-Fuller Test on Residuals of Equation (12)

Test Specification	Test Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Trend	-0.689	-3.99	-3.43	-3.13	1
Drift	-0.714	-3.46	-2.88	-2.57	1
None	-0.742	-2.58	-1.95	-1.62	1

the case; I would have thought that the estimated coefficients might be a renormalization of those in the regression of Equation (11).[7]

Putting aside my apparent lack of understanding of the mechanics behind cointegration and regression, my conclusion remains that I would propose the modification given by Equation (11) to the demand for money, given that I do expect there to be cointegration vectors based on the Johansen results and therefore an ECM representation for the model.

[7]It might be the case that the regression estimated is spurious, that the supposed evidence of cointegration is due to any of the arguments proposed against the residual based tests in the style of Engle and Granger (1987) by Phillips and Ouliaris (1990, p. 166).

3. Markov Switching Models

Problem 3. Use the EViews data file `argentinaactual.wf1` The file contains the following variables: `pbireal` (for real GDP) and 30 and 60-day interest rates. Identify the booms and recessions of the Argentinean economy using

(a) A Markov Switching Model using the growth rate. NB: Only assume that the unconditional mean switches.
(b) A Markov Switching Model using the growth rate. NB: Only assume that the unconditional mean switches. Include the (lagged) spread of interest rates as a probability regressor.
(c) A Markov Switching Model using the growth rate. NB: Only assume that the unconditional mean and the unconditional variance switch. Include the (lagged) spread as a probability regressor.
(d) Interpret and compare the results obtained in (a), (b), and (c).

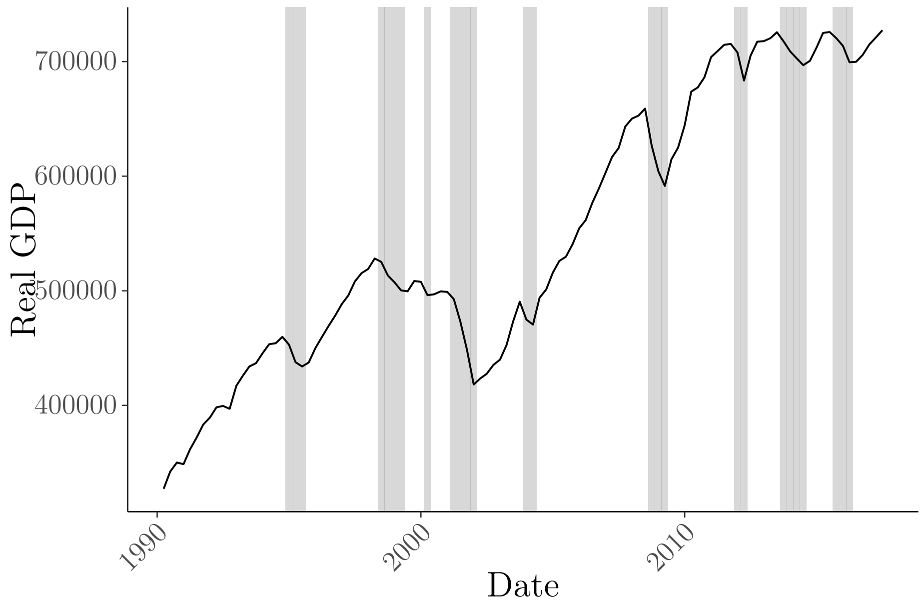
Solution 3 (a). I attempted to estimate the Markov Switching models of Problem 3 using R and Python, in an effort to maintain reproducibility, but the packages available were unfortunately unsatisfactory. The results that follow were consequently obtained using the EViews GUI, and no code is provided for this solution (although `code/prob_3.py` and `code/prob_3.R` contain my initial attempts). Details on the procedure provided by the EViews output are, however, provided alongside the tables in order to make clearer the steps taken.

We first estimate a Markov Switching model using the growth rate, assuming that only the unconditional mean switches. The estimation results can be found in Table 39. The series for real GDP, with recessions shaded according to the model, is shown in Figure 16. The series for GDP, with a gradient that indicates the smoothed recession probability according to the model, is shown in Figure 16b. Similar to Hamilton (1989, p. 374), in which a similar model is estimated, we find that the particular binary classification rule based upon the smoothed probabilities (classify as recession if $P(\text{recession}) > 0.5$) is not particularly relevant; few observations are assigned intermediate probabilities. This is also visually true of the models estimated in solutions 3 (b) and 3 (c).

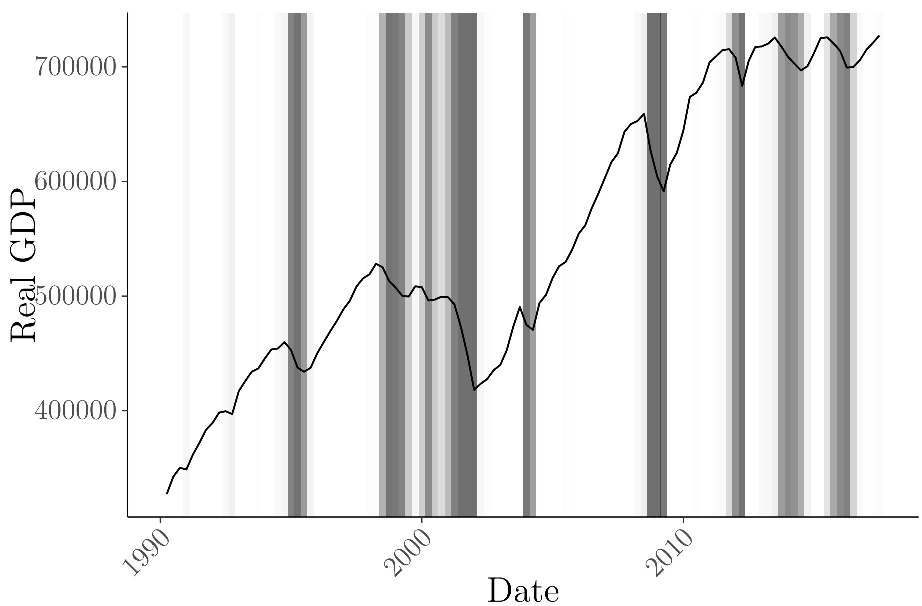
Solution 3 (b). We now turn to a model in which lagged interest rate spreads are included as a probability regressor. The estimation results can be found in Table 40. The series for real GDP, with recessions shaded according to the model, is shown in Figure 17.

Solution 3 (c). Finally, we estimate a Markov Switching model that allows both the unconditional mean and variance to switch, including the lagged spread as a probability regressor. The estimation results can be found in Table 41. The series for GDP growth, with recessions shaded according to the model, is shown in Figure 18. The series for GDP growth, with a gradient that indicates the smoothed recession probability according to the model, is shown in Figure 18.

Solution 3 (d). We are now asked to comment on the results of the estimated models. Let us first note that all three models seem to match relatively well with "known" recessions, such as 1998-2002 "Argentine great depression". Moreover, there is not a substantial difference between models in the periods that are classified as recessions. The three Markov Switching models differ in complexity, with Model A obviously being the most simple and Model C



(A) Real GDP with Recessions (Model A)



(B) Real GDP with Recession Probability (Model A)

FIGURE 16. Real GDP with Recession Indicators (Model A)

the most complex. Which is preferred will depend on the purpose of the model, and the researcher's attitude towards overfitting and parsimony.

Model A, which allows only for shifts in the unconditional mean of GDP growth, captures the basic distinction between recession and expansion. Regime 1 corresponds to a boom, with a growth rate of 0.0119, while Regime 2 corresponds to a recession, with a growth rate of -0.0378. The effect of the constant on switching probabilities is highly statistically significant. If one is conservative with respect to overfitting and values parsimony, its performance

Table 39. Simple Switching Regression Results: Specification A

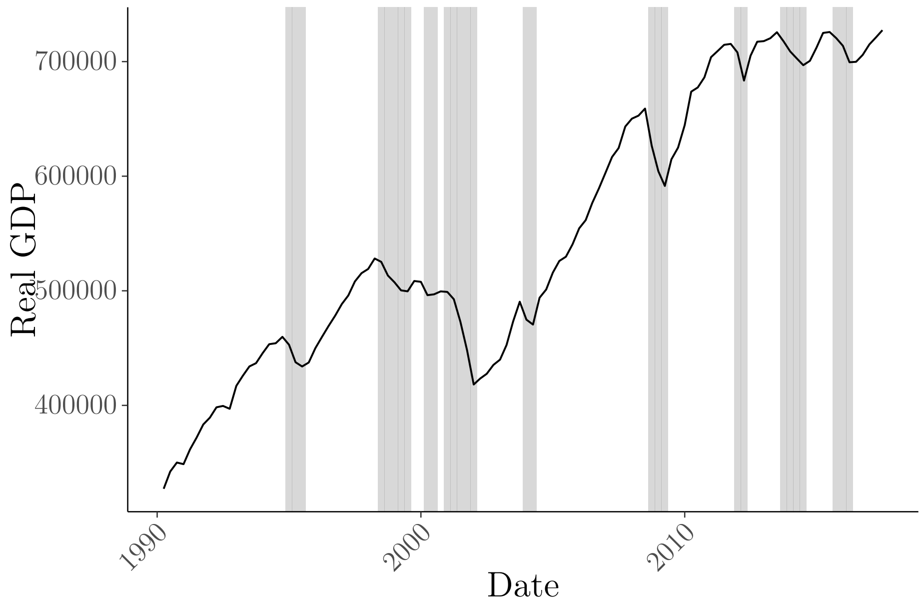
Variable	Coefficient	Std. Error	z-Statistic	Prob.
Regime 1				
`C`	0.011854	0.001912	6.199624	0.0000
Regime 2				
`C`	−0.037790	0.008122	−4.652598	0.0000
Common				
`LOG(SIGMA)`	−4.132767	0.084924	−48.66449	0.0000
Probabilities Parameters				
`P1-C`	2.317888	0.487208	4.757495	0.0000
Model Statistics				
Mean dependent var	0.007404	S.D. dependent var		0.021508
S.E. of regression	0.021708	Sum squared resid		0.050421
Durbin-Watson stat	1.055125	Log likelihood		273.9377
Akaike info criterion	−4.907958	Schwarz criterion		−4.809759
Hannan-Quinn criter.	−4.868128			

Dependent Variable: `PBI_G`
Method: Simple Switching Regression (BFGS / Marquardt steps)
Date: 06/26/25 Time: 23:19
Sample (adjusted): 1990Q2 2017Q3
Included observations: 110 after adjustments
Number of states: 2
Standard errors & covariance computed using observed Hessian
Random search: 25 starting values with 10 iterations using 1 standard deviation (rng=tm, seed=195514018)
Convergence achieved after 7 iterations

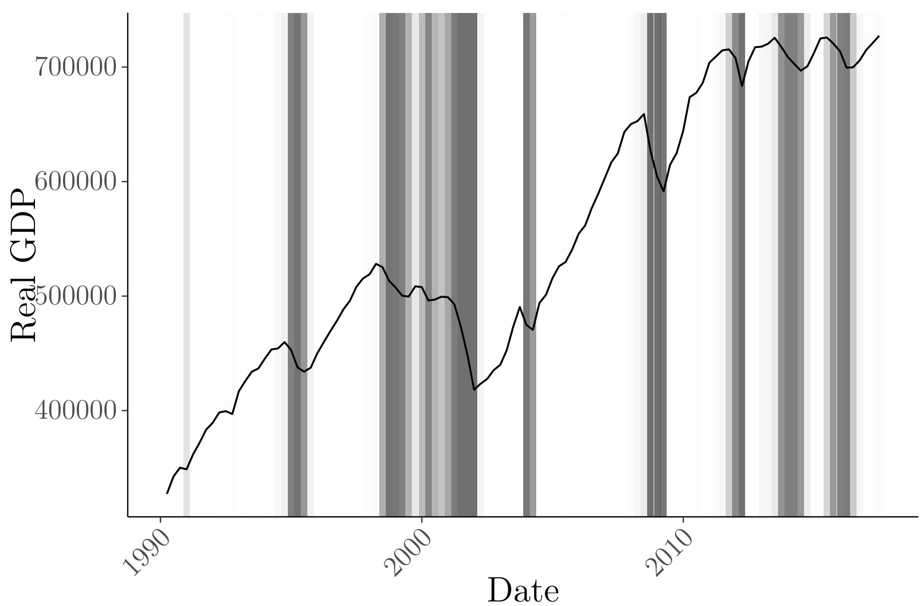
is not terrible, as judged by the recessions that it fails to identify if compared with the other models. As noted in Problem (a) and shown in Figure 16, most observations are assigned relatively extreme probabilities. The log-likelihood is the worst out of the three models, and the AIC puts it in second place.

Model B incorporates the lagged interest rate spread into the transition probabilities, introducing a plausible leading indicator. The recession and boom regimes are respectively Regime 1 and Regime 2, with growth rates of −0.0080 and 0.0190. The spread has a statistically significant (albeit modest) effect on the persistence of the high-growth regime (`P11`), suggesting that a steeper curve marginally increases the probability of remaining in expansion. However, its influence on transitions out of recession (`P21`) is not statistically significant. The model reduces the standard error of the regression and improves the likelihood marginally, but the improvement in information criteria over Model A is negligible, possibly suggesting overfitting.

Model C further extends the model by allowing the variance to switch across regimes. The inclusion of this feature may be motivated by the fact that recessions are empirically associated with greater volatility. The estimation confirms this: Regime 2, interpreted as recession, exhibits a significantly higher standard deviation. Nevertheless, the spread loses



(A) Real GDP with Recessions (Model B)



(B) Real GDP with Recession Probability (Model B)

Figure 17. Real GDP with Recession Indicators (Model B)

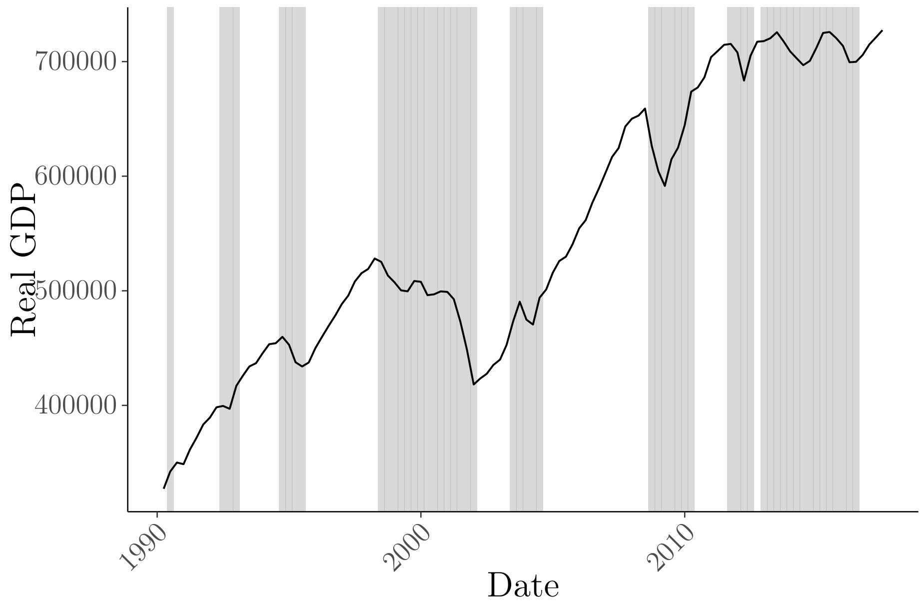
all explanatory power in the transition equations: none of the coefficients are significant (although the effect on `P11` is rejected at 5% by a somewhat small margin), which implies that once conditional heteroskedasticity is permitted, the spread no longer contributes meaningfully to predicting regime shifts. This makes sense; in Model B, the spread might have been a proxy for volatility. Model C yields the highest likelihood and the best AIC and HQIC values, but the Schwarz criterion (BIC) mildly favors Model A, in line with its characteristically more parsimonious selections.

Table 40. Markov Switching Regression Results: Specification B

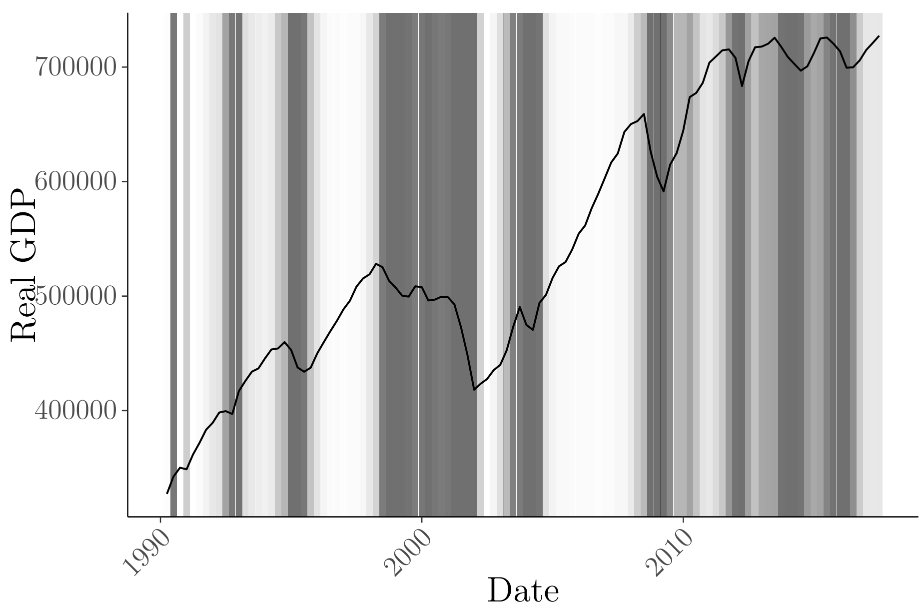
Variable	Coefficient	Std. Error	z-Statistic	Prob.
Regime 1				
C	−0.007989	0.003903	−2.047060	0.0407
Regime 2				
C	0.019016	0.002632	7.223469	0.0000
Common				
LOG(SIGMA)	−4.091016	0.095074	−43.02983	0.0000
Transition Matrix Parameters				
P11-C	1.938514	0.739236	2.622322	0.0087
P11-SPREAD(-1)	0.256015	0.129110	1.982925	0.0474
P21-C	−1.559041	0.568988	−2.740023	0.0061
P21-SPREAD(-1)	−1.082521	1.070993	−1.010764	0.3121
Model Statistics				
Mean dependent var	0.007404	S.D. dependent var		0.021508
S.E. of regression	0.019587	Sum squared resid		0.041052
Durbin-Watson stat	1.497974	Log likelihood		276.8658
Akaike info criterion	−4.906651	Schwarz criterion		−4.734802
Hannan-Quinn criter.	−4.836948			

Dependent Variable: PBI_G
Method: Markov Switching Regression (BFGS / Marquardt steps)
Sample (adjusted): 1990Q2 2017Q3
Included observations: 110 after adjustments
Number of states: 2
Initial probabilities obtained from ergodic solution
Standard errors & covariance computed using observed Hessian
Random search: 25 starting values with 1 iteration using 1 standard deviation (rng=km, seed=26769242)
Convergence achieved after 16 iterations

We should also note that as we introduce complexity, the models progressively identify a larger fraction of the sample with recession. Whether this undesired, if the purpose of the model is classification, will depend on what we want to describe as a recession.



(A) Real GDP with Recessions (Model C)



(B) Real GDP with Recession Probability (Model C)

FIGURE 18. Real GDP with Recession Indicators (Model C)

Table 41. Markov Switching Regression: Specification C

Variable	Coefficient	Std. Error	z-Statistic	Prob.
Regime 1				
`C`	0.018750	0.001932	9.704202	0.0000
`LOG(SIGMA)`	−4.413579	0.104232	−42.34388	0.0000
Regime 2				
`C`	−0.010274	0.005610	−1.831295	0.0671
`LOG(SIGMA)`	−3.874853	0.145280	−26.67157	0.0000
Transition Matrix Parameters				
`P11-C`	2.044714	0.475358	4.301421	0.0000
`P11-SPREAD(-1)`	0.010658	0.052432	0.203273	0.8389
`P21-C`	−1.332736	0.817749	−1.629761	0.1032
`P21-SPREAD(-1)`	−0.425542	0.638890	−0.666064	0.5054
Model Statistics				
Mean dependent var	0.007404	S.D. dependent var		0.021508
S.E. of regression	0.019815	Sum squared resid		0.041621
Durbin-Watson stat	1.458503	Log likelihood		284.8050
Akaike info criterion	−5.032818	Schwarz criterion		−4.836420
Hannan-Quinn criter.	−4.953158			

Dependent Variable: `PBI_G`
Method: Markov Switching Regression (BFGS / Marquardt steps)
Sample (adjusted): 1990Q2 2017Q3
Included observations: 110 after adjustments
Number of states: 2
Initial probabilities obtained from ergodic solution
Standard errors & covariance computed using observed Hessian
Random search: 25 starting values with 1 iteration using 1 standard deviation (rng=km, seed=26769242)
Convergence achieved after 16 iterations

4. Bonds

Problem 4. Use the EViews data file `AnnLee.wf1`. The file contains the variables `EX3MHOLD12`, `EX3MHOLD24`, `EX3MHOLD60` and `EX3MHOLD120`, which represents the excess (with respect to the 3 months rate) realized return of holding 3 months a bond of maturity 12, 24, 60 and 120.

(a) Use the Kalman filter to extract and store a common factor that explains the movements of those returns.
(b) Use the common factor stored before to assess whether the slope and curvature are variables with explanatory power to explain those (average) returns.

Solution 4 (a). The dataset provided contains variables `EX3MHOLD12`, `EX3MHOLD24`, `EX3MHOLD60` and `EX3MHOLD120`, which represent the returns over a three month period for bonds with maturities of 12, 24, 60, and 120 months, respectively, in excess of the 3-month rate. Let us call these variables R_{12}, R_{24}, R_{60}, and R_{120}, for brevity. The state-space model is motivated theoretically by the idea that there is a latent *common factor* that affects the excess returns of the different bonds in ways that are different for each bond, but constant in time (and linear in the parameter to be estimated). We will later provide a potential interpretation of this common factor, which we will denote as x_t. Suppose the scalar x_t is governed by the AR(1) process

$$x_t = Fx_{t-1} + v_t, \tag{13}$$

where $v_t \sim \mathcal{N}(0, \sigma_v^2)$ is a white noise process. This is called the *state equation* and it is unobservable by nature. The excess returns are called *signals* in the nomenclature of the model, are observable, and are described by

$$R_{n,t} = \lambda_n x_t + \epsilon_{n,t}, \quad n = 12, 24, 60, 120, \tag{14}$$

where λ_n is the so-called *factor loading* for the bond of maturity n and $\epsilon_{n,t} \sim \mathcal{N}(0, \sigma_n^2)$ is a white noise process, independent of the other residual or error terms. We can sum up the model in matrix form as

$$\mathbf{R}_t = \boldsymbol{\lambda} x_t + \boldsymbol{\epsilon}_t, \tag{15}$$

where

$$\mathbf{R}_t = \begin{bmatrix} R_{12,t} \\ R_{24,t} \\ R_{60,t} \\ R_{120,t} \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_{12} \\ \lambda_{24} \\ \lambda_{60} \\ \lambda_{120} \end{bmatrix}, \quad \boldsymbol{\epsilon}_t = \begin{bmatrix} \epsilon_{12,t} \\ \epsilon_{24,t} \\ \epsilon_{60,t} \\ \epsilon_{120,t} \end{bmatrix}.$$

What is to be estimated, then, are the parameters F, σ_v^2, $\boldsymbol{\lambda}$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{12}^2 & 0 & 0 & 0 \\ 0 & \sigma_{24}^2 & 0 & 0 \\ 0 & 0 & \sigma_{60}^2 & 0 \\ 0 & 0 & 0 & \sigma_{120}^2 \end{bmatrix}.$$

As in Problem 3, an attempt was made to implement the Kalman filter using open-source languages like Python and R, to no avail. The implementation was therefore carried out in EViews, wherein certain steps are not documented, because of the role of the GUI, but the specification of the state space object used for estimation is the following:[8]

[8]When I first estimated this model, I had set certain initial values in the environment during previous experimentation, which lead to convergence on the first attempt. I later restarted the session and was unable

```
@signal EX3MHOLD12 = sv1  + [var = exp(c(5))]
@signal EX3MHOLD24 = c(2)*sv1  + [var = exp(c(6))]
@signal EX3MHOLD60 = c(3)*sv1  + [var = exp(c(7))]
@signal EX3MHOLD120 = c(4)*sv1 + [var = exp(c(8))]

@state sv1 = c(9)*sv1(-1) + [var = exp(c(10))]

param c(2) 2.128982
param c(3) 4.370830
param c(4) 6.971379
param c(5) -0.314057
param c(6) 0.0
param c(7) 3.013411
param c(8) 5.002321
param c(9) 0.705313
param c(10) 1.265331
```

Note the normalization of $\lambda_{12} = 1$ in the first signal equation. I am not entirely sure whether this is necessary or not. When I first attempted to estimate the model, including λ_{12} as a parameter to be estimated, I could not get the estimation to converge (or even run sometimes), but when I normalized it to 1, the procedure converged immediately. That made me suspect that the normalization was necessary, possibly due to the system not being identified otherwise, or collinearity issues. Later, though, I was able to estimate models without the normalization, albeit with collinearity issues. I am not sure whether that is a consequence of not normalizing or a numerical coincidence. I stand by the normalization, in any case, because it gives the common factor x_t in units of the excess return of the 12-month bond.

The results of the estimation are shown in Table 42. What we are most interested in are the estimated factor loadings λ_n.[9] The key result is that they are all of the same sign (in particular, positive) and increasing in the maturity of the bond. An account of the relationships captured by the model can be thought of as follows.

It is an empirical fact with strong microeconomic foundations that investors demand a higher return from riskier assets. The bonds considered are (assumed to be) issued by the same entity, and therefore the heterogeneity in returns can be attributed principally to the larger volatility inherent in the prediction of the price over a longer horizon.[10]

---

to estimate the model without setting initial values for parameters, so I estimated it again using the specific values shown in the code. The seed is set with the command `rndseed 091218`.

[9]Though we should make note of the massive standard error on the variance of the 24-month bond, the cause of which I am not sure of. Although the estimation converged, the might be overfitting or numerically ill-conditioned. Since the coefficients of interest make sense, I am not inclined to dwell on it.

[10]More specifically, the price within three months of a bond with a longer maturity should be less predictable than the price within three months of a bond with a shorter maturity, all else equal, because the bond price three months from now must incorporate the expectation of the evolution of the price over the remaining time to maturity, which will be longer for the longer bond, and thus more sensitive to changes in price over the three months (under weak assumptions; e.g., Martingale properties). This characteristic is often referred to as *price risk* by practitioners. F. J. Fabozzi and F. A. Fabozzi (2000, p. 214) is a classic primer on the categorization of the types of risk affecting bond investment.

TABLE 42. Kalman Filter Estimation: State Space Model

Variable	Coefficient	Std. Error	z-Statistic	Prob.
Estimated Coefficients				
λ_{24}	2.128982	0.018801	113.2376	0.0000
λ_{60}	4.370830	0.069333	63.04158	0.0000
λ_{120}	6.971379	0.186124	37.45555	0.0000
$\log \sigma_{12}^2$	−0.314057	0.044190	−7.107016	0.0000
$\log \sigma_{24}^2$	−24.78705	4730000000.0	−0.00000000524	1.0000
$\log \sigma_{60}^2$	3.013411	0.059342	50.78018	0.0000
$\log \sigma_{120}^2$	5.002321	0.055318	90.42856	0.0000
F	0.705313	0.015523	45.43658	0.0000
$\log \sigma_v^2$	1.265331	0.034073	37.13626	0.0000
Final State				
x	−0.466020	1.882622	−0.247538	0.8045
Model Statistics				
Log likelihood	−7553.456			
Akaike info criterion	21.85681			
Schwarz criterion	21.91585			
Hannan-Quinn criter.	21.87964			

Dependent Variables: `EX3MHOLD12, EX3MHOLD24, EX3MHOLD60, EX3MHOLD120`
Method: Maximum likelihood (BFGS / Marquardt steps)
Sample: 1962M01 2019M11
Included observations: 695
Valid observations: 692
Convergence achieved after 46 iterations
Coefficient covariance computed using outer product of gradients

We can think of the common factor x_t as a measure of the *risk-premium* associated with bonds of longer maturity, and in particular normalized to units of excess return of the 12-month bond over the three month rate. For example, if

$$E_{t-1}R_{12,t} = 1,$$

then the estimated value of the risk premium for the 24-month bond is

$$E_{t-1}R_{24,t} = 2.12,$$

which means that the 12-month bond is expected to yield 1 percentage point in excess of the interest rate over the three month period, while the 24-month bond is expected to yield 2.12 percentage points in excess of the interest rate over the same period, which is 2.12 units of the excess return on the 12-month bond.

The fact that coefficients are increasing in maturity is consistent with our interpretation of the common factor as a risk premium. When the risk premium demanded by investors is higher, the required return increases more for the longer bonds than for the shorter ones. When risk premium falls, the required return falls more for the longer bonds than for the shorter ones.

We can also note that the standard errors of the coefficients are monotone increasing in maturity, which is consistent with the higher volatility of the longer bonds' returns; the way in which their expected returns react to changes in the common factor is less stable. This characteristic of the data is reflected visually in Figure 19, which overlays all the "ex post yield curves" constructed by connecting the points $(n, R_{n,t})$, $n = 12, 24, 60, 120$ for each month t in the sample.

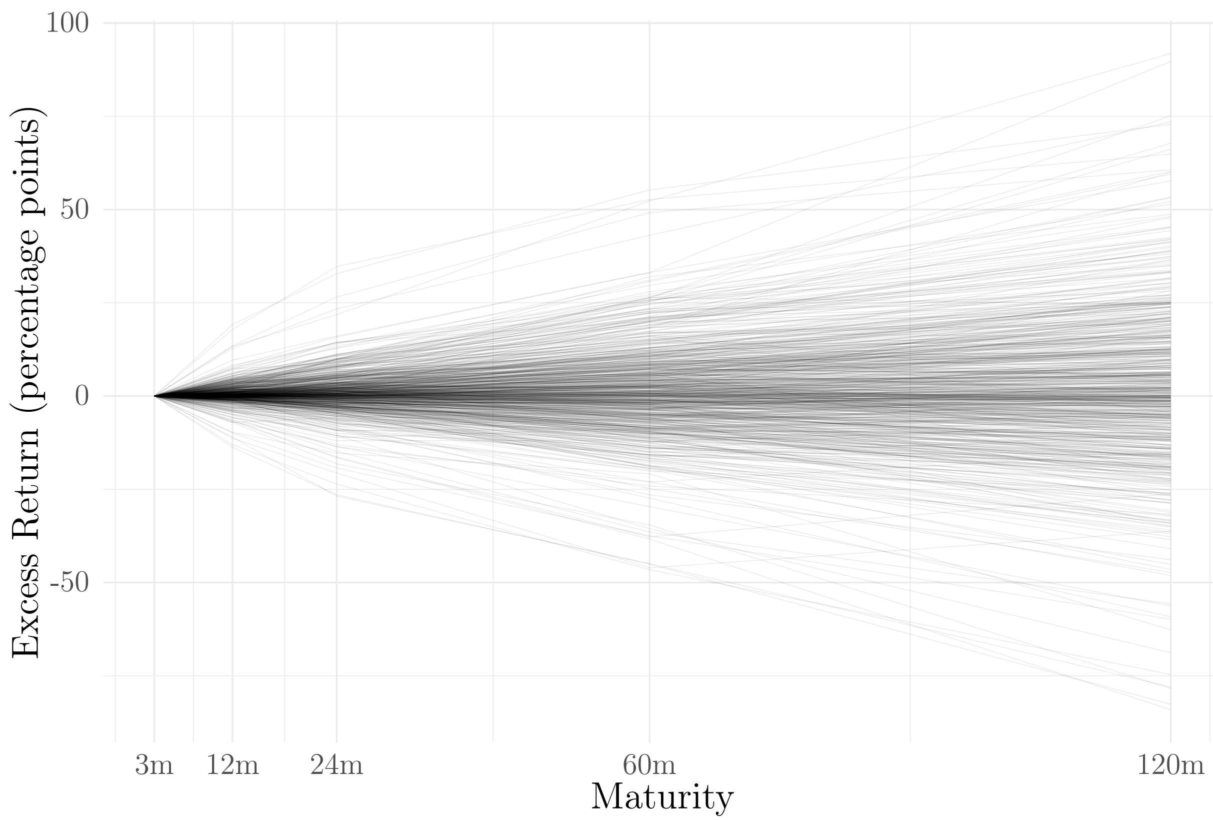


FIGURE 19. Ex post yield curves

At this point in the analysis, it seems like the latent factor is telling us about relative reactions of yields of different maturities on the curve to changes in investor attitudes, part of which is intermingled with the slope. We analyze the relationship with slope and curvature of the term structure (yield curve) in detail in the next problem.

Solution 4 (b). The common factor series $\mathtt{x}_t$ used in the following analysis is the *smoothed state estimate*. The variables $\mathtt{slope}_t$ and $\mathtt{curvature}_t$ represent the slope and curvature of the yield curve. Histograms of the three series are displayed in Figure 20. The results of the estimation of

$$\mathtt{x}_t = \beta_0 + \beta_1 \mathtt{slope}_t + \beta_2 \mathtt{curvature}_t + u_t \tag{16}$$

are displayed in Table 43. The results show that all parameter estimates of the regression are statistically significant individually and jointly, with very small p-values. The coefficient associated with the slope of the yield curve is positive and the one associated with the curvature is negative. These results indicate that both slope and curvature possess statistically significant explanatory power for the common factor x_t, which, as argued above, can be

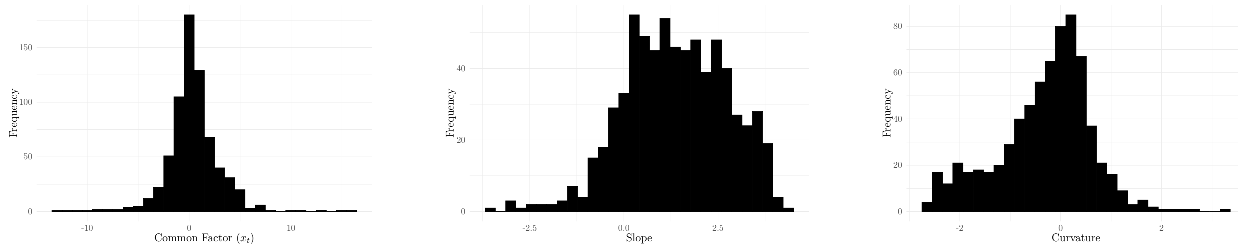
TABLE 43. Regression of Common Factor on Slope and Curvature

	Dependent variable:
	x_t
Slope	0.589
	(0.075)***
	p = 0.000
Curvature	−0.288
	(0.110)***
	p = 0.010
Constant	−0.514
	(0.133)***
	p = 0.0002
Observations	692
R^2	0.127
Adjusted R^2	0.125
Residual Std. Error	2.462 (df = 689)
F Statistic	50.318*** (df = 2; 689)
Note:	*p<0.1; **p<0.05; ***p<0.01

interpreted as a risk premium or term premium factor. The estimated coefficient on the slope variable is positive, which is consistent with the common macro-finance interpretation of a steep yield curve as signaling elevated expected excess returns on longer bonds, typically occurring during early expansions or recoveries, when investors demand higher compensation for bearing interest rate risk. And we can think of the the negative effect of curvature on the common factor as resulting from the idea that curvature tends to peak around business cycle turning points, which can be though of as periods in which medium-term expectations are elevated due to transient macroeconomic uncertainty or policy changes, but in which investors do not yet demand substantially higher compensation for long-term risks. In such periods, the term premium is compressed even as medium yields rise, reflecting temporary distortions rather than sustained risk compensation.

The common factor x_t compresses the two separate effects of slope and curvature into one. It is likely, then, that the estimation of a state-space model with more latent factors, such as in Diebold and Li (2006), would allow one to disentangle the effects of slope and curvature by means of latent factors which are interpreted to have a correspondence with the two characteristics.

5. Markov Chain Monte Carlo



(A) Common Factor (B) Slope (C) Curvature

FIGURE 20. Histograms of Common Factor, Slope, and Curvature

Problem 5. Use the EViews data file `Shiller2018a.wf1`. Estimate for the following periods: 1900-1929, 1945-1987 and 1945-2018, the equation:

$$P_t = \alpha_0 + \alpha_1 D_t + \varepsilon_t \tag{17}$$

Where `rsp`, P_t, represents real stock prices and `rdiv`, D_t, real dividends. NB: Estimate the model using Markov Chain Monte Carlo techniques. HINT: You must adapt the routine `gbs_ar4:chapter7` by Kim and Nelson.

(a) Notice that real stock prices are the expected discount value of future dividends, that is:

$$P_t = \mathbb{E}_t \sum_{i=1}^{\infty} \left[\frac{1}{1+r}\right]^i D_{t+i} \tag{18}$$

Show that if we assume that D_t follows a random walk with drift, equation (18) can be written as equation (17). What are exactly α_0 and α_1 ? HINT: You should use the following expressions:

$$\sum_{s=1}^{\infty} a^s = \frac{a}{1-a}, \quad \sum_{s=1}^{\infty} a^s s = \frac{a}{(1-a)^2}$$

Is there any evidence that D_t follows a Random Walk in any subsamples?

(b) Report your results for the whole sample for equation (17).

(c) Compare the dispersion of α_1 obtained using the results of the full sample with those obtained using the sub-samples listed above. Comment on the results. NB: When modifying the routine you need to take into account

i. That there are two regressors instead of 5 as in `gbs_ar4`.

ii. That you should remove the control

```
COEF = -REV(BETA_F[2:5])|1;
ROOT = POLYROOT(COEF);
ROOTMOD = ABS(ROOT);

IF MINC(ROOTMOD) GE 1.0001;
    ACCEPT = 1;
ELSE;
    ACCEPT = 0;
ENDIF;
```

Solution 5 (a). We are asked to show that under the assumption that dividends follow a random walk with drift, the infinite sum representation of stock prices can be written as the linear relationship given in the first equation. Under the random walk with drift assumption, we have

$$D_t = \mu + D_{t-1} + \eta_t, \tag{19}$$

where μ is the drift parameter and η_t is white noise. This specification implies that $\mathbb{E}_t[D_{t+i}] = D_t + i\mu$. Substituting into the fundamental valuation equation, we obtain

$$P_t = \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i (D_t + i\mu) = D_t \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i + \mu \sum_{i=1}^{\infty} i \left(\frac{1}{1+r}\right)^i. \tag{20}$$

Applying the identities from Equation (18) with $a = 1/(1+r)$, we have

$$\sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i = \frac{1}{r} \quad \text{and} \quad \sum_{i=1}^{\infty} i \left(\frac{1}{1+r}\right)^i = \frac{1+r}{r^2}. \tag{21}$$

Substituting these results into Equation (20) yields

$$P_t = \frac{\mu(1+r)}{r^2} + \frac{1}{r} D_t. \tag{22}$$

Comparing this with the regression specification $P_t = \alpha_0 + \alpha_1 D_t + \varepsilon_t$, we identify

$$\alpha_0 = \frac{\mu(1+r)}{r^2}, \tag{23}$$

$$\alpha_1 = \frac{1}{r}. \tag{24}$$

The intercept α_0 captures the present value of expected dividend growth due to the drift, while the slope coefficient α_1 represents the price-dividend ratio under constant discount rate assumptions. Let us note that $\alpha_1 = 1/r$ implies that the discount rate can be estimated as $r = 1/\alpha_1$, provided the random walk with drift assumption holds.

We use the ADF testing procedure described in the solution to Problem 2 (and later implemented as well in Problem 6) to test for the presence of a unit root in the real dividends series, D_t in each subsample and obtain the results in Tables 44, 45 and 46. Based on an

Table 44. ADF Test Results for Real Dividends (1900-1929)

Specification	Variable	Test Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Trend	D_t (Real Dividends)	−1.593	−4.15	−3.50	−3.18	1
Drift	D_t (Real Dividends)	−1.820	−3.58	−2.93	−2.60	1
None	D_t (Real Dividends)	0.179	−2.62	−1.95	−1.61	1

examination of the results, which we will not describe in detail because a similar analysis is performed in both Problem 2 and Problem 6, we may conclude that the evidence seems to suggest that dividends follow a random walk in the first and last period, while evidence is mixed in the middle period, in which we might have rejected the null hypothesis of a unit root upon reaching the second stage wherein we test the `drift` specification of the ADF test (and would have rejected if our level were 5% but not if it were 1%).

Table 45. ADF Test Results for Real Dividends (1945-1987)

Specification	Variable	Test Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Trend	D_t (Real Dividends)	−2.686	−4.15	−3.50	−3.18	1
Drift	D_t (Real Dividends)	−3.103	−3.58	−2.93	−2.60	1
None	D_t (Real Dividends)	0.710	−2.62	−1.95	−1.61	1

Table 46. ADF Test Results for Real Dividends (1945-2018)

Specification	Variable	Test Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Trend	D_t (Real Dividends)	−0.314	−4.04	−3.45	−3.15	1
Drift	D_t (Real Dividends)	0.993	−3.51	−2.89	−2.58	1
None	D_t (Real Dividends)	2.126	−2.60	−1.95	−1.61	1

Solution 5 (b). The results of the Gibbs Sampler approach to the estimation of the model in Equation (17) for the full sample are shown in Table **??**.

Table 47. Posterior Results for Full Sample (1900–2018): $P_t = \alpha_0 + \alpha_1 D_t$

Parameter	Mean	Std. Dev.	Median
α_0	−1.8419	0.1970	−1.8418
α_1	61.3948	2.4890	61.4076
σ^2	0.9690	0.1297	0.9605

Table 48. Comparison of α_1 Dispersion

Period	α_1 Mean	α_1 Std. Dev.
Full Sample (1900–2018)	61.3948	2.4890
Pre-Depression (1900–1929)	21.7310	4.4427
Post-War I (1945–1987)	40.3730	4.2908
Post-War II (1945–2018)	64.9557	3.8602

Solution 5 (c). We can carry out a comparison of α_1 estimates across periods in an informal way to think about why we might have gotten these results. The full sample estimate benefits from the largest sample size, which naturally leads to the lowest standard deviation (2.49) among all estimates. This supports the basic fact that more observations provide information for parameter identification, as one would expect from basic sampling theory.

The pre-Depression period (1900–1929) shows the lowest α_1 estimate at 21.73, implying a relatively high discount rate of approximately 4.6%. This seems plausible given the higher real interest rates that characterized the early twentieth century. The higher standard deviation (4.44) likely reflects both the smaller sample size and probably also the greater volatility in market conditions during this period.

The post-war periods show intermediate values, with the Post-War I period (1945–1987) yielding $\alpha_1 = 40.37$ and the extended Post-War II period (1945–2018) giving $\alpha_1 = 64.96$. The fact that the latter is quite close to the full sample estimate suggests that the post-1945 period dominates the overall relationship, which does makes sense, given that this represents the majority of the sample. The standard deviations for these subperiods (4.29 and 3.86, respectively) are higher than the full sample but lower than the pre-Depression period, which is consistent with the trade-off between sample size and parameter uncertainty.

6. Sustainability of Trade Deficits

Problem 6. Use the EViews data file `data_impexp.wf1`. The file has data on imports, exports, prices and GDP for USA. The variable `tbovergdp`, represents the trade balance over GDP. Consider the period 1947Q1-2005Q1.

(a) Evaluate, at 2005Q1, the sustainability of the trade deficits using the econometric methods seen in the course.

Solution 6 (a). The typical analysis of the sustainability of deficits (trade, fiscal, etc.) involves testing whether the inflows (in our case exports) and the outflows (in our case imports) are cointegrated. Notable works include Husted (1992), Hakkio and Rush (1991), and Quintos (1995).

The methods used in this problem are introduced in the solutions to Problem 2. What we will first do is find the order of integration of the series for real exports and real imports. As the analysis specifies that we must evaluate the sustainability of the trade deficits "at 2005Q1",[11] we will not use data after that date in forming our conclusions. Following the procedure outlined in Problem 2, we first run the so-called `trend` specification of the Augmented Dickey-Fuller (ADF) test on variables `rexp` and `rimp`, which represent, respectively, real exports and real imports, as deflated by the consumer price index (`cpi`).[12]

Table 49 shows the results of the ADF test on each series.

Table 49. Augmented Dickey-Fuller Test Results (Trend Specification)

Variable	τ_τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
`rexp`	-2.299	-3.99	-3.43	-3.13	1
`rimp`	-0.908	-3.99	-3.43	-3.13	1

Since we do not reject the null hypothesis of a unit root, we proceed to carry out the `drift` specification of the ADF test. Results for this test can be seen in Table 50.

Table 50. Augmented Dickey-Fuller Test Results (Drift Specification)

Variable	τ_μ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
`rexp`	0.485	-3.46	-2.88	-2.57	1
`rimp`	1.940	-3.46	-2.88	-2.57	1

We once again fail to reject the null hypothesis of a unit root. Finally, we run the `none` specification of the ADF test, the results of which are displayed in Table 51.

Again, under this specification we fail to reject, and so we may conclude that there is a unit root.

[11] I was unsure whether this was a typo, since data was provided up to 20**2**5Q1, but since "2005Q1" is repeated twice in the problem statement, I will assume it is not.

[12] In obtaining series for real gdp, since the price index series contains many missing values near the start of the sample, our effective sample becomes 1955Q1-2005Q1.

Table 51. Augmented Dickey-Fuller Test Results (None Specification)

Variable	τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
`rexp`	2.860	-2.58	-1.95	-1.62	1
`rimp`	3.634	-2.58	-1.95	-1.62	1

We now test the differenced series for stationarity. The results of the `trend` specification of the ADF test on the first difference of each series can be seen in Table 52.

Table 52. Augmented Dickey-Fuller Test Results (Trend Specification, Differenced Series)

Variable	τ_τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
`drexp`	-6.623	-3.99	-3.43	-3.13	1
`drimp`	-7.303	-3.99	-3.43	-3.13	1

Clearly we may reject the null hypothesis of a unit root in both series, and so we conclude that both `rexp` and `rimp` are integrated of order one, $I(1)$.

Once again following the methodology of Engle and Granger (1987), we now regress one vector on the other, in order to test whether the residuals of the regression are $I(0)$, meaning that the regression estimate is a cointegration vector and can be interpreted as a long-run relationship.

We estimate the regression of real exports on real imports, the OLS estimates of which are shown in Table 53.

Table 53. Regression of Real Exports on Real Imports

	Dependent variable:
	`rexp`
`rimp`	0.698^{***}
	(0.011)
Constant	1.305^{***}
	(0.116)
Observations	201
R^2	0.952
Adjusted R^2	0.951
Residual Std. Error	0.949 (df = 199)
F Statistic	$3{,}904.599^{***}$ (df = 1; 199)
Note:	*p<0.1; **p<0.05; ***p<0.01

We also estimate the regression of real imports on real exports, the OLS estimates of which are shown in Table 54.

Table 54. Regression of Real Imports on Real Exports

	Dependent variable:
	`rimp`
`rexp`	1.364^{***}
	(0.022)
Constant	-1.369^{***}
	(0.183)
Observations	201
R^2	0.952
Adjusted R^2	0.951
Residual Std. Error	1.327 (df = 199)
F Statistic	$3{,}904.599^{***}$ (df = 1; 199)
Note:	*p<0.1; **p<0.05; ***p<0.01

For both the regressions, we follow the procedure for testing for unit roots by way of successively nested ADF tests. We begin with the `trend` specification, the results of which are shown for both regressions in Table 55.

We fail to reject the null hypothesis of a unit root in the residuals of both regressions, meaning that we now proceed to the `drift` specification of the ADF test, the results of which are shown in Table 56.

Again we fail to reject, so we carry out the ADF test with the `none` specification, the results of which are shown in Table 57.

Table 55. Augmented Dickey-Fuller Test on Residuals (Trend Specification)

Variable	τ_τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Residuals (`rexp` on `rimp`)	−0.314	−3.99	−3.43	−3.13	1
Residuals (`rimp` on `rexp`)	−0.204	−3.99	−3.43	−3.13	1

Table 56. Augmented Dickey-Fuller Test on Residuals (Drift Specification)

Variable	τ_μ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Residuals (`rexp` on `rimp`)	−0.717	−3.46	−2.88	−2.57	1
Residuals (`rimp` on `rexp`)	−0.205	−3.46	−2.88	−2.57	1

TABLE 57. Augmented Dickey-Fuller Test on Residuals (None Specification)

Variable	τ Statistic	Critical Values 1%	5%	10%	Lags (AIC)
Residuals (`rexp` on `rimp`)	-0.733	-2.58	-1.95	-1.62	1
Residuals (`rimp` on `rexp`)	-0.234	-2.58	-1.95	-1.62	1

Since we fail to reject the null hypothesis of a unit root in the residuals of both regressions, for neither of the two can we say that the residuals are $I(0)$, and therefore neither coefficient vector can be interpreted as a cointegration vector.

Before concluding that the series do not cointegrate, which is a necessary condition for sustainability of the trade deficit (Husted, 1992), we will also test for cointegration using the Johansen test, which is typically considered a better test than the easier to use ADF test (Phillips and Ouliaris, 1990).

The optimal lag is selected by AIC, and the results of the Johansen test for cointegration are shown in Table 58. Since we do not reject the null hypothesis of no cointegration, we

TABLE 58. Johansen Cointegration Test Results

Hypothesis	Test Statistic	Critical Values 10%	5%	1%
$r \leq 1$	3.216	7.52	9.24	12.97
$r = 0$	16.206	17.85	19.96	24.60

may be more assured in our conclusion from the previous Engle-Granger methodology that there is imports and exports do not seem to cointegrate.

Our final conclusion is thus that the behavior of imports and exports in the sample analyzed does not suggest that the series cointegrate, and therefore we cannot conclude that the trade deficit is sustainable.

References

Bollerslev, Tim (Apr. 1, 1986). "Generalized Autoregressive Conditional Heteroskedasticity". In: *Journal of Econometrics* 31.3, pp. 307–327. ISSN: 0304-4076. DOI: 10.1016/0304-4076(86)90063-1. URL: https://www.sciencedirect.com/science/article/pii/0304407686900631 (visited on 06/16/2025).

Bollerslev, Tim, Robert F. Engle, and Jeffrey M. Wooldridge (1988). "A Capital Asset Pricing Model with Time-Varying Covariances". In: *Journal of Political Economy* 96.1, pp. 116–131. ISSN: 0022-3808. JSTOR: 1830713. URL: https://www.jstor.org/stable/1830713 (visited on 06/17/2025).

Cont, R. (Feb. 1, 2001). "Empirical Properties of Asset Returns: Stylized Facts and Statistical Issues". In: *Quantitative Finance* 1.2, pp. 223–236. ISSN: 1469-7688. DOI: 10.1080/713665670. URL: https://doi.org/10.1080/713665670 (visited on 06/17/2025).

Dickey, David A. and Wayne A. Fuller (1979). "Distribution of the Estimators for Autoregressive Time Series With a Unit Root". In: *Journal of the American Statistical Association* 74.366, pp. 427–431. ISSN: 0162-1459. DOI: 10.2307/2286348. JSTOR: 2286348. URL: https://www.jstor.org/stable/2286348 (visited on 06/23/2025).

– (1981). "Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root". In: *Econometrica* 49.4, pp. 1057–1072. ISSN: 0012-9682. DOI: 10.2307/1912517. JSTOR: 1912517. URL: https://www.jstor.org/stable/1912517 (visited on 06/24/2025).

Diebold, Francis X. and Canlin Li (Feb. 1, 2006). "Forecasting the Term Structure of Government Bond Yields". In: *Journal of Econometrics* 130.2, pp. 337–364. ISSN: 0304-4076. DOI: 10.1016/j.jeconom.2005.03.005. URL: https://www.sciencedirect.com/science/article/pii/S0304407605000795 (visited on 06/27/2025).

Engle, Robert F. (1982). "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation". In: *Econometrica* 50.4, pp. 987–1007. ISSN: 0012-9682. DOI: 10.2307/1912773. JSTOR: 1912773. URL: https://www.jstor.org/stable/1912773 (visited on 06/16/2025).

Engle, Robert F. and C. W. J. Granger (1987). "Co-Integration and Error Correction: Representation, Estimation, and Testing". In: *Econometrica* 55.2, pp. 251–276. ISSN: 0012-9682. DOI: 10.2307/1913236. JSTOR: 1913236. URL: https://www.jstor.org/stable/1913236 (visited on 06/25/2025).

Fabozzi, Frank J. and Francesco A. Fabozzi (2000). *Bond Markets, Analysis, and Strategies, Tenth Edition*. Bloomington, Ind.: The MIT Press. 936 pp. ISBN: 978-0-253-33753-5.

Fuller, Wayne A. (1976). *Introduction to Statistical Time Series*. Wiley. 488 pp. ISBN: 978-0-471-28715-5. Google Books: pYwpAQAAMAAJ.

Hakkio, Craig S. and Mark Rush (1991). "Is the Budget Deficit "Too Large?"" In: *Economic Inquiry* 29.3, pp. 429–445. ISSN: 1465-7295. DOI: 10.1111/j.1465-7295.1991.tb00837.x. URL: https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1465-7295.1991.tb00837.x (visited on 06/27/2025).

Hamilton, James D. (1989). "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle". In: *Econometrica* 57.2, pp. 357–384. ISSN: 0012-9682. DOI: 10.2307/1912559. JSTOR: 1912559. URL: https://www.jstor.org/stable/1912559 (visited on 06/27/2025).

– (1994). *Time Series Analysis*. Princeton (N.J.): Princeton University Press. ISBN: 978-0-691-04289-3.

Hansen, Peter R. and Asger Lunde (2005). "A Forecast Comparison of Volatility Models: Does Anything Beat a GARCH(1,1)?" In: *Journal of Applied Econometrics* 20.7, pp. 873–889. ISSN: 1099-1255. DOI: 10.1002/jae.800. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/jae.800 (visited on 06/17/2025).

Husted, Steven (1992). "The Emerging U.S. Current Account Deficit in the 1980s: A Cointegration Analysis". In: *The Review of Economics and Statistics* 74.1, pp. 159–166. ISSN: 0034-6535. DOI: 10.2307/2109554. JSTOR: 2109554. URL: https://www.jstor.org/stable/2109554 (visited on 06/27/2025).

Johansen, Søren (1991). "Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models". In: *Econometrica* 59.6, pp. 1551–1580. ISSN: 0012-9682. DOI: 10.2307/2938278. JSTOR: 2938278. URL: https://www.jstor.org/stable/2938278 (visited on 06/24/2025).

Ljung, G. M. and G. E. P. Box (Aug. 1, 1978). "On a Measure of Lack of Fit in Time Series Models". In: *Biometrika* 65.2, pp. 297–303. ISSN: 0006-3444. DOI: 10.1093/biomet/65.2.297. URL: https://doi.org/10.1093/biomet/65.2.297 (visited on 06/17/2025).

Nelson, Charles R. and Charles R. Plosser (Jan. 1, 1982). "Trends and Random Walks in Macroeconomic Time Series: Some Evidence and Implications". In: *Journal of Monetary Economics* 10.2, pp. 139–162. ISSN: 0304-3932. DOI: 10.1016/0304-3932(82)90012-5. URL: https://www.sciencedirect.com/science/article/pii/0304393282900125 (visited on 06/25/2025).

Nelson, Daniel B. (1991). "Conditional Heteroskedasticity in Asset Returns: A New Approach". In: *Econometrica* 59.2, pp. 347–370. ISSN: 0012-9682. DOI: 10.2307/2938260. JSTOR: 2938260. URL: https://www.jstor.org/stable/2938260 (visited on 06/17/2025).

Pfaff, Bernhard (2008). *Analysis of Integrated and Cointegrated Time Series with R*. New York, NY: Springer. ISBN: 978-0-387-75966-1 978-0-387-75967-8. DOI: 10.1007/978-0-387-75967-8. URL: http://link.springer.com/10.1007/978-0-387-75967-8 (visited on 06/24/2025).

Phillips, P. C. B. and S. Ouliaris (1990). "Asymptotic Properties of Residual Based Tests for Cointegration". In: *Econometrica* 58.1, pp. 165–193. ISSN: 0012-9682. DOI: 10.2307/2938339. JSTOR: 2938339. URL: https://www.jstor.org/stable/2938339 (visited on 06/27/2025).

Quintos, Carmela E. (1995). "Sustainability of the Deficit Process with Structural Shifts". In: *Journal of Business & Economic Statistics* 13.4, pp. 409–417. ISSN: 0735-0015. DOI: 10.2307/1392386. JSTOR: 1392386. URL: https://www.jstor.org/stable/1392386 (visited on 06/27/2025).

Solá, Martín (2025a). *ARCH Models*. Lecture notes. Universidad Torcuato Di Tella. URL: https://acrobat.adobe.com/id/urn:aaid:sc:VA6C2:e8e34d11-89ea-4951-aff4-24af3f1b1801.

– (Mar. 2025b). *Non-Stationary Stochastic Time Series Models*. Presentation Slides. URL: https://acrobat.adobe.com/link/review?uri=urn%3Aaaid%3Ascds%3AUS%3A84538430-d120-38a6-8573-b015d30c062d.

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