A COUNTEREXAMPLE TO A CONJECTURE ON NAKAMURA NUMBERS FOR WEIGHTED GAMES

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1. Introduction

This note provides a counterexample to the upper bound proposed in Conjecture 2 of Freixas and Kurz (2019, p. 626). For every integer $n \geq 5$, we construct a weighted game with n voters, five types, no vetoers, and Nakamura number $n-2$, which exceeds the proposed bound of $n-3$.

2. Definitions

In order to state the conjecture to which the counterexample pertains, some definitions are required.

Definition 1 (Simple game)**.** Let $N = \{1, \ldots, n\}$, with $n \geq 2$, denote the set of voters. A coalition is a set $S \subset N$.[1] The set of winning coalitions is a set $\mathcal{W} \subset 2^N$, where 2^N is the set of subsets of N. A *simple game* on N is a pair $G = (N, \mathcal{W})$ satisfying $\emptyset \notin \mathcal{W}$ and $N \in \mathcal{W}$, and satisfying the following property: if $S, R \in 2^N$, $S \subset R$, and $S \in \mathcal{W}$, then $R \in \mathcal{W}$.

Definition 2 (Weighted game)**.** A *weighted game* is a simple game such that there exist weights $w_i \geq 0$ and a quota q with $0 < q \leq \sum_{i \in N} w_i$ such that for all $S \subset N$, $S \in \mathcal{W}$ if and only if $\sum_{i \in S} w_i \geq q$. We denote a given representation of a weighted game by $[q; w_1, w_2, \ldots, w_n]$, and the class of weighted games by $\mathcal{T}$. It is also convenient to define $w(S) = \sum_{i \in S} w_i$ for $S \subset N$.

Definition 3 (Vetoer)**.** Given the simple game $G = (N, \mathcal{W})$, a player $i \in N$ is a *vetoer* if $i \in S$ for every $S \in \mathcal{W}$.

It follows immediately from Definition 3 that a game G has no vetoers if and only if $\cap_{S \in \mathcal{W}} S = \emptyset$.

Definition 4 (Nakamura number)**.** Given a simple game $G = (N, \mathcal{W})$, the *Nakamura number* of G is defined as

$$\nu(G) = \min\left\{ k \in \mathbb{N}_{>0} : \exists S_1, \ldots, S_k \in \mathcal{W}, \bigcap_{i=1}^{k} S_i = \emptyset \right\},$$

if $\cap_{S \in \mathcal{W}} S = \emptyset$, and $\nu(G) = \infty$ otherwise.[2]

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[1] Weak set inclusion is denoted by $\subset$, allowing equality.

[2] The set $\mathbb{N}_{>0}$ is $\{k \in \mathbb{N} : k > 0\}$.

Definition 5 (Voter types)**.** Given a game $G = (N, \mathcal{W})$, voters $i, j \in N$ are *equivalent*, denoted $i \sim_G j$, if for every $S \subset N \setminus \{i, j\}$, we have $S \cup \{i\} \in \mathcal{W}$ if and only if $S \cup \{j\} \in \mathcal{W}$. The induced equivalence classes $[i]_G = \{j \in N : j \sim_G i\}$ are called types, and the number of types for a game is denoted $t(G) = |\{[i]_G : i \in N\}|$.

Definition 6. Let n and t be integers with $n \geq 2$ and $1 \leq t \leq n$, and let $N = \{1, \ldots, n\}$. The set of weighted games on N with exactly t types and no vetoers is

$$\mathcal{T}(n,t) = \left\{ G = (N, \mathcal{W}) \in \mathcal{T} : t(G) = t, \bigcap_{S \in \mathcal{W}} S = \emptyset \right\}.$$

If $\mathcal{T}(n,t) \neq \emptyset$, the largest Nakamura number among the games in this set is defined as

$$\mathrm{Nak}^{\mathcal{T}}(n,t) = \max_{G \in \mathcal{T}(n,t)} \nu(G).$$

3. Conjecture

Freixas and Kurz (2019) conjectured the following bounds on the maximum Nakamura number of weighted games.

Conjecture 1. *If n is sufficiently large, we have $n - t + 1 \leq \mathrm{Nak}^{\mathcal{T}}(n,t) \leq n - t + 2$, where $t \in \mathbb{N}_{>0}$.*

The next section disproves Conjecture 1.

4. Counterexample

Consider the family of weighted games

$$G_n = [5n - 16; 5^{(n-4)}, 4, 3, 2, 1]$$

indexed by the integer $n \geq 5$.[3] We shall show that for every integer $n \geq 5$, the game G_n belongs to $\mathcal{T}(n, 5)$ and has Nakamura number $n - 2$, contradicting the conjectured upper bound of $n - 3$. We first state a preliminary lemma.

Lemma 1. *Let $G = (N, \mathcal{W})$ be a weighted game and $i, j, k \in N$ distinct voters satisfying $w_i \geq w_j$ and $w_j \geq w_k$. If $i \sim_G k$, then $i \sim_G j$.*

Proof. Take $S \subset N \setminus \{i, j\}$. If $S \cup \{j\} \in \mathcal{W}$, then $S \cup \{i\} \in \mathcal{W}$ because $w_i \geq w_j$. Conversely, suppose $S \cup \{i\} \in \mathcal{W}$. If $k \notin S$, then $i \sim_G k$ implies $S \cup \{k\} \in \mathcal{W}$, and since $w_j \geq w_k$, we have $S \cup \{j\} \in \mathcal{W}$. If $k \in S$, then, defining $R = S \setminus \{k\}$, we have $R \cup \{i, k\} \in \mathcal{W}$. Since $R \cup \{i, k\} = S \cup \{i\}$, and $w_j \geq w_k$, we have $R \cup \{i, j\} \in \mathcal{W}$. Because $i \sim_G k$, we have $R \cup \{k, j\} \in \mathcal{W}$. But $R \cup \{k, j\} = S \cup \{j\}$, which shows that $S \cup \{j\} \in \mathcal{W}$, and therefore $i \sim_G j$. □

We may now prove Proposition 1, which will contradict the conjectured upper bound.

[3] The term $5^{(n-4)}$ denotes that the first $n - 4$ voters in the game have weight 5.

Proposition 1. *For every integer $n \geq 5$, the weighted game with n voters*

$$G_n = [5n - 16; 5^{(n-4)}, 4, 3, 2, 1]$$

has no vetoers, exactly 5 types, and Nakamura number $n - 2$.

Proof. Take G_n as defined above, letting $q = 5n - 16$ denote the quota and w_i denote the weight of voter i. The total weight is $w(N) = 5n - 10$, and for every $S \subset N$, we have $w(N \setminus S) = w(N) - w(S)$. From the definition of a weighted game, $S \in \mathcal{W}$ if and only if $w(S) \geq 5n - 16$, if and only if $w(N \setminus S) \leq 6$. For every $i \in N$, we have $w(\{i\}) = w_i \leq 5$, which implies that $N \setminus \{i\} \in \mathcal{W}$. Therefore $\cap_{S \in \mathcal{W}} S = \emptyset$ and G_n has no vetoers.

Next, we show that G_n has exactly 5 types of voters. It is immediate that voters with equal weights are equivalent, because if $w_i = w_j$, then $w(S \cup \{i\}) = w(S \cup \{j\})$ for any $S \subset N \setminus \{i, j\}$ and therefore $S \cup \{i\} \in \mathcal{W}$ if and only if $S \cup \{j\} \in \mathcal{W}$, meaning $i \sim_{G_n} j$. Thus the game G_n can have at most 5 types. For each weight $r \in \{1, 2, 3, 4, 5\}$, there exists a voter $p_r \in N$ with weight $w_{p_r} = r$. Given voters p_1, p_2, p_3, p_4, p_5, we define the set

$$H = N \setminus \{p_1, p_2, p_3, p_4, p_5\},$$

which consists of the $n - 5$ remaining voters, each with weight 5, so $w(H) = 5(n - 5)$. Define the coalitions

$$\begin{aligned} S_5 &= H \cup \{p_3, p_1\}, \\ S_4 &= H \cup \{p_5\}, \\ S_3 &= H \cup \{p_5, p_1\}, \\ S_2 &= H \cup \{p_4, p_3\}. \end{aligned}$$

For every $r \in \{2, 3, 4, 5\}$, the coalition S_r excludes voters p_r and p_{r-1}, and $w(S_r) = q - r$. It follows that $w(S_r \cup \{p_r\}) = q$ and $w(S_r \cup \{p_{r-1}\}) = q - 1$, which means that $S_r \cup \{p_r\} \in \mathcal{W}$ and $S_r \cup \{p_{r-1}\} \notin \mathcal{W}$. Thus $p_r \nsim_{G_n} p_{r-1}$ for every $r \in \{2, 3, 4, 5\}$. To show that no pair of voters with different weights is equivalent, suppose for the sake of contradiction that $p_r \sim_{G_n} p_s$ for some $r > s$. If $r = s + 1$, we arrive at a contradiction with $p_r \nsim_{G_n} p_{r-1}$. Otherwise, $r \geq s + 2$, and we have three distinct voters p_r, p_{r-1}, and p_s with weights $w_{p_r} > w_{p_{r-1}}$ and $w_{p_{r-1}} > w_{p_s}$, which by Lemma 1 implies that $p_r \sim_{G_n} p_{r-1}$, a contradiction. Therefore no pair of the voters p_1, p_2, p_3, p_4, and p_5 is equivalent. Every voter $i \in H$ has the same weight as p_5 (namely, 5), and therefore $i \sim_{G_n} p_5$. Since $N = H \cup \{p_1, p_2, p_3, p_4, p_5\}$, the types for the game G_n are the equivalence classes

$$\begin{aligned} [p_r]_{G_n} &= \{p_r\}, \quad r \in \{1, 2, 3, 4\}, \\ [p_5]_{G_n} &= H \cup \{p_5\}. \end{aligned}$$

Thus $t(G_n) = 5$.

It remains to show that $\nu(G_n) = n-2$. Let $S_1, \ldots, S_k \in \mathcal{W}$ with $k \in \mathbb{N}_{>0}$ be a collection of coalitions satisfying

$$\bigcap_{\ell=1}^{k} S_\ell = \emptyset.$$

We established above that a coalition S belongs to $\mathcal{W}$ if and only if $w(N \setminus S) \leq 6$, therefore, defining

$$B_\ell = N \setminus S_\ell, \quad \ell \in \{1, \ldots, k\},$$

each B_ℓ satisfies $w(B_\ell) \leq 6$. Furthermore, $\cup_{\ell=1}^{k} B_\ell = N$, because for each voter $i \in N$, there exists $\ell \in \{1, \ldots, k\}$ such that $i \notin S_\ell$ (and hence $i \in B_\ell$), otherwise the intersection of $S_1, \ldots, S_k$ would not be empty. Thus the Nakamura number is the least number of subsets of N, each with weight at most 6, with union N. There are $n-4$ voters i with weight $w_i = 5$, and for each exists at least one $\ell \in \{1, \ldots, k\}$ such that $i \in B_\ell$. Moreover, no set B_ℓ can contain two voters of weight 5 because $w(B_\ell) \leq 6$. Therefore at least $n-4$ of the sets $B_1, \ldots, B_k$ contain a voter of weight 5. Using $w(B_\ell) \leq 6$, none of the coalitions containing a voter of weight 5 can contain p_2, p_3, or p_4, since the resulting weight of the coalition would exceed 6. Thus voters p_2, p_3, and p_4 must belong to additional coalitions. Because $w(\{p_2, p_3, p_4\}) = 9$, it also must be the case that these three voters belong to at least two additional sets. Counting the $n-4$ sets of voters with weight 5 along with the minimum of two additional sets, we must have $k \geq n-2$. Since $S_1, \ldots, S_k$ were taken to be an arbitrary collection of winning coalitions with empty intersection, we have shown that the Nakamura number for G_n satisfies

$$\nu(G_n) \geq n-2. \tag{1}$$

We now show that the Nakamura number of G_n is exactly $n-2$, by constructing a collection of $n-2$ winning coalitions with empty intersection. Consider the collection

$$\mathcal{B} = \{\{p_5, p_1\}, \{p_4, p_2\}, \{p_3\}\} \cup \{\{i\} : i \in H\}.$$

The first three sets have weights $w(\{p_5, p_1\}) = 6$, $w(\{p_4, p_2\}) = 6$, and $w(\{p_3\}) = 3$, and because $w(\{i\}) = 5$ for each $i \in H$, we have $w(B) \leq 6$ for each $B \in \mathcal{B}$. The union of the sets is $\cup_{B \in \mathcal{B}} B = N$, and because $|H| = n-5$, the collection $\mathcal{B}$ contains exactly $n-2$ sets. Using $w(B) \leq 6$, $N \setminus B \in \mathcal{W}$ for every $B \in \mathcal{B}$, and

$$\bigcap_{B \in \mathcal{B}} (N \setminus B) = \emptyset$$

because $\cup_{B \in \mathcal{B}} B = N$. Therefore, using equation 1, we have

$$\nu(G_n) = n-2,$$

which concludes the proof. □

Proposition 1 establishes that for every integer $n \geq 5$, $\nu(G_n) = n - 2$, and $G_n \in \mathcal{T}(n, 5)$. By the definition of $\text{Nak}^{\mathcal{T}}(n, 5)$, it follows that

$$\text{Nak}^{\mathcal{T}}(n, 5) \geq n - 2,$$

but Conjecture 1 asserts that for sufficiently large n

$$\text{Nak}^{\mathcal{T}}(n, 5) \leq n - 3.$$

Since $\nu(G_n) = n - 2$ holds for every integer $n \geq 5$, the conjecture is false.

References

Freixas, Josep and Sascha Kurz (2019). "Bounds for the Nakamura Number." In: *Social Choice and Welfare* 52.4, pp. 607–634. DOI: 10.1007/s00355-018-1164-y.